

ALGEBRAIC COBORDISM THEORY ATTACHED TO ALGEBRAIC EQUIVALENCE

AMALENDU KRISHNA AND JINHYUN PARK

ABSTRACT. Based on the algebraic cobordism theory of Levine and Morel, we develop a theory of algebraic cobordism modulo algebraic equivalence.

We prove that this theory can reproduce Chow groups modulo algebraic equivalence and the semi-topological K_0 -groups. We also show that with finite coefficients, this theory agrees with the algebraic cobordism theory.

We compute our cobordism theory for some low dimensional varieties. The results on infinite generation of some Griffiths groups by Clemens and on smash-nilpotence by Voevodsky and Voisin are also lifted and reinterpreted in terms of this cobordism theory.

1. INTRODUCTION

The theory of algebraic cobordism Ω^* was developed by Levine and Morel [12]. This theory is modelled on the geometric description of complex cobordism theory MU^* by Quillen [16]. The most interesting aspect of the algebraic cobordism theory is that it is universal among the oriented cohomology theories on smooth algebraic varieties. One of the consequences of this universality is that algebraic cobordism contains enough data to reproduce Chow groups and Grothendieck groups of algebraic varieties, as shown in [12, Theorem 4.5.1, Corollary 4.2.12] (see also [2]). This is in contrast with the topological situation where the natural map $MU^*(X) \otimes_{\mathbb{L}^*} \mathbb{Z} \rightarrow H^*(X, \mathbb{Z})$ is known to be not an isomorphism in general for a CW-complex X (see [18, Theorem 2.2]).

More recently, Levine and Pandharipande [13] defined the double-point cobordism theory ω_* and showed that this is isomorphic to the Levine-Morel algebraic cobordism theory Ω_* . As a consequence, the artificially imposed formal group law in the Levine-Morel definition of Ω_* gets a geometric interpretation. For a scheme X over a field k , $\omega_*(X)$ is defined in terms of cobordism cycles over X and the equivalence relation between these cobordism cycles is given in terms of a family of smooth cobordism cycles over $X \times \mathbb{P}^1$ which degenerate to a simple normal crossing of two smooth divisors in the family. The precise definition will be recalled below. It was remarked by Levine and Pandharipande (see [13, § 11.2]) that a double point cobordism theory based on algebraic equivalence should exist if one considers families of cobordism cycles parameterized by more general curves than just \mathbb{P}^1 .

This leads one to the following natural question: is there a theory of algebraic cobordism based on the Levine-Morel model, which reflects *algebraic equivalence*, reproduces Chow groups modulo algebraic equivalence and the semi-topological Grothendieck group, and more generally, interpolates between the algebraic and the complex cobordism? The goal of this paper is to develop such a theory. We go on to show that this new cobordism theory based on algebraic equivalence, also recovers the one suggested by Levine-Pandharipande in *op.cit.* This was one of our motivations which led to the genesis of this paper.

2010 *Mathematics Subject Classification.* Primary 14F43; Secondary 55N22.

Key words and phrases. cobordism, Chow group, K-theory, algebraic cycle, Griffiths group.

We show that the algebraic cobordism theory Ω_*^{alg} interpolates between the algebraic and the complex cobordism in much the same way the semi-topological K -theory of Friedlander, Lawson and Walker (see [3], [4], [5]) interpolates between the algebraic and the topological K -theories. We compute Ω_*^{alg} for curves and surfaces and show that they are finitely generated modules over the Lazard ring. This is in contrast with the corresponding situation in algebraic cobordism. We further show that Ω_*^{alg} agrees with Ω_* with finite coefficients. Our hope is that the functor Ω_*^{alg} will inherit the properties of algebraic as well as the complex cobordism. As a consequence, this may be better suited for the study of complex algebraic varieties.

Let k always denote a fixed base field of characteristic zero throughout the paper. A *scheme* in this paper will mean a separated scheme of finite type over k . We shall denote the category of schemes by \mathbf{Sch}_k . The full subcategory of smooth quasi-projective schemes over k will be denoted by \mathbf{Sm}_k . In this paper, an oriented Borel-Moore homology theory on \mathbf{Sch}_k and an oriented cohomology theory on \mathbf{Sm}_k will mean the ones considered in [12, Definitions 5.1.3, 1.1.2]. Let \mathbb{L}_* denote the Lazard ring (see [12, p. 4]).

The central results of this paper can be summarized as follows:

Theorem 1.1. *On the category \mathbf{Sch}_k , there are two isomorphic algebraic cobordism theories attached to algebraic equivalence : Ω_*^{alg} in Definition 3.4 and ω_*^{alg} in Definition 4.3, which satisfy the following properties:*

(1) *The functor Ω_*^{alg} defines an oriented Borel-Moore homology theory on \mathbf{Sch}_k that respects algebraic equivalence in the sense of Definition 3.9. The restriction Ω_{alg}^* on the subcategory \mathbf{Sm}_k , with the cohomological indexing in Definition 3.4, defines an oriented cohomology theory that respects algebraic equivalence in the sense of Definition 3.9.*

(2) *Ω_*^{alg} satisfies the localization property, the \mathbb{A}^1 -homotopy invariance and the projective bundle formula.*

(3) *Ω_{alg}^* is universal among all oriented cohomology theories on \mathbf{Sm}_k that respect algebraic equivalence. Similarly, Ω_*^{alg} is universal among all oriented Borel-Moore homology theories on \mathbf{Sch}_k that respect algebraic equivalence.*

(4) *For $X \in \mathbf{Sch}_k$, $\Omega_*^{\text{alg}}(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \simeq \text{CH}_*^{\text{alg}}(X)$ and $\Omega_*^{\text{alg}}(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}] \simeq G_0^{\text{semi}}(X)[\beta, \beta^{-1}]$, where CH_*^{alg} is the Chow group modulo algebraic equivalence, G_0^{semi} is the semi-topological Grothendieck group of coherent sheaves, and β is a formal symbol of degree -1 .*

(5) *For $X \in \mathbf{Sch}_k$, $\Omega_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m \xrightarrow{\sim} \Omega_*^{\text{alg}}(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m$.*

Theorem 1.2. *Let $X \in \mathbf{Sch}_k$ and let \mathbb{L}^* be the Lazard ring with the cohomological indexing.*

(1) *For X smooth over \mathbb{C} , there is a cycle class map $\Omega_{\text{alg}}^*(X) \rightarrow \text{MU}^{2*}(X(\mathbb{C}))$.*

(2) *$\mathbb{L}^* \simeq \Omega^*(k) \simeq \Omega_{\text{alg}}^*(k)$. Furthermore, Ω_*^{alg} is generically constant in the sense of [12, Definition 4.1.1].*

(3) *If X is a cellular scheme, then $\Omega_*(X) \xrightarrow{\sim} \Omega_*^{\text{alg}}(X)$ as free \mathbb{L}_* -modules.*

(4) *When X is smooth, the \mathbb{L}^* -module $\Omega_{\text{alg}}^*(X)$ is finitely generated if and only if the group $\text{CH}_{\text{alg}}^*(X)$ is finitely generated. When X is smooth projective over \mathbb{C} , the \mathbb{L}^* -module $\Omega_{\text{alg}}^*(X)$ is finitely generated if and only if the Griffiths group $\text{Griff}^*(X)$ is finitely generated. If $\dim X \leq 2$, the \mathbb{L}^* -module $\Omega_{\text{alg}}^*(X)$ is finitely generated, and it is not necessarily finitely generated otherwise.*

(5) *If X is a connected smooth affine curve, then $\mathbb{L}^* \simeq \Omega_{\text{alg}}^*(X)$. If X is a smooth curve over \mathbb{C} , then $\Omega_{\text{alg}}^*(X) \xrightarrow{\sim} \text{MU}^{2*}(X(\mathbb{C}))$ and an analogue of Quillen-Lichtenbaum conjecture holds:*

$$\Omega^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m \xrightarrow{\sim} \text{MU}^{2*}(X(\mathbb{C})) \otimes_{\mathbb{Z}} \mathbb{Z}/m.$$

The following result of this paper is a cobordism analogue of a theorem of Voevodsky [20] and Voisin [21] about smash-nilpotence for algebraic cycles.

Theorem 1.3. *Let X be a smooth projective scheme and α be a cobordism cycle over X . If α vanishes in $\Omega_{\text{alg}}^*(X)_{\mathbb{Q}}$, then its smash-product $\alpha^{\boxtimes N}$ (see Definition 10.1) vanishes in $\Omega^*(X^N)_{\mathbb{Q}}$ for some integer $N > 0$.*

The organization of this paper is as follows. A good part of the paper between § 2 and § 8 is devoted to proving Theorem 1.1. § 2 recalls the definition of cobordism cycles from [12], and that of algebraic equivalence. In § 3, we define Ω_*^{alg} in terms of the cobordism cycles of Levine-Morel modulo various relations, one of which reflects algebraic equivalence of line bundles. We prove a universal property of Ω_*^{alg} . § 4 recalls the rational and algebraic double-point cobordism theories ω_* and ω_*^{alg} from [13].

Our main step in proving many of the above results is the basic exact sequence of Theorem 5.1. This sequence gives an explicit relation between $\omega_*(X)$ and $\omega_*^{\text{alg}}(X)$ for any $X \in \mathbf{Sch}_k$. This in particular allows us to prove the isomorphism between $\Omega_*^{\text{alg}}(X)$ and $\omega_*^{\text{alg}}(X)$ and Theorem 1.1(2) in § 6. § 7 finishes the proof of Theorem 1.1(1)(3) and § 8 proves Theorem 1.1(4)(5). In § 9, we compute Ω_*^{alg} from various angles and prove Theorem 1.2. In § 10, we discuss a cobordism analogue of smash-nilpotence of algebraically trivial algebraic cycles and prove Theorem 1.3.

Some details related to Gysin pull-backs from [12] are placed in the Appendix (§ 11), and there a new lemma related to Ω_*^{alg} is proven.

This paper builds on two grand works [12] and [13] on algebraic cobordism. Whenever necessary, we take the freedom of using the definitions and results of these references. In doing so, we will provide full reference details. When no confusion arises, we shall use \sim to mean algebraic equivalence.

2. COBORDISM CYCLES AND ALGEBRAIC EQUIVALENCE

This section recalls the basic definitions in the theory of algebraic cobordism of Levine and Morel [12]. We also recall the notion of algebraic equivalence of vector bundles and algebraic cycles. These will be used in the construction of our cobordism theory in § 3.

2.1. Cobordism cycles. Recall the following from [12, Definition 2.1.6]:

Definition 2.1. Let $X \in \mathbf{Sch}_k$ be of dimension $n \geq 0$. An *integral cobordism cycle over X* is a collection $(f: Y \rightarrow X, L_1, \dots, L_r)$, where Y is smooth and integral, f is projective, and L_1, \dots, L_r ($r \geq 0$) are line bundles on Y . Its dimension is defined to be $\dim(Y) - r \in \mathbb{Z}$. An *isomorphism between two cobordism cycles* $(Y \rightarrow X, L_1, \dots, L_r) \xrightarrow{\sim} (Y' \rightarrow X, L'_1, \dots, L'_{r'})$ is a triple $\Phi = (\phi: Y \rightarrow Y', \sigma, (\psi_1, \dots, \psi_r))$ consisting of an isomorphism $\phi: Y \rightarrow Y'$ of X -schemes, a bijection $\sigma: \{1, \dots, r\} \simeq \{1, \dots, r'\}$, and isomorphisms $\psi_i: L_i \simeq \phi^* L'_{\sigma(i)}$ of lines bundles over Y for all i . When Y is a smooth scheme with several components, define $(Y \rightarrow X, L_1, \dots, L_r)$ to be the sum of the obvious integral cobordism cycles corresponding to the components.

Let $\mathcal{Z}_*(X)$ be the free abelian group on the set of isomorphism classes of integral cobordism cycles over X . We let $\mathcal{Z}_d(X)$ be the subgroup generated by the dimension d cobordism cycles. The image of the integral cobordism cycle $(Y \rightarrow X, L_1, \dots, L_r)$ in $\mathcal{Z}_*(X)$ is denoted by $[Y \rightarrow X, L_1, \dots, L_r]$. When X is smooth and equidimensional, the class $[\text{Id}_X: X \rightarrow X] \in \mathcal{Z}_d(X)$ is denoted by 1_X . A cobordism cycle of the form $[\text{Id}_X: X \rightarrow X, L_1, \dots, L_r]$ is often written as $[X \rightarrow X, L_1, \dots, L_r]$ when no confusion arises. Recall the following definitions from [12, 2.1.2, 2.1.3]:

Definition 2.2. (1) For a projective morphism $g: X \rightarrow X'$ in \mathbf{Sch}_k , the *push-forward along g* is the graded group homomorphism $g_*: \mathcal{Z}_*(X) \rightarrow \mathcal{Z}_*(X')$ given by the composition with g , that is, $[f: Y \rightarrow X, L_1, \dots, L_r] \mapsto [g \circ f: Y \rightarrow X', L_1, \dots, L_r]$.

(2) For a smooth equidimensional morphism $g: X \rightarrow X'$ of relative dimension d , the *pull-back along g* is the homomorphism $g^*: \mathcal{Z}_*(X') \rightarrow \mathcal{Z}_{*+d}(X)$ given by sending $[f: Y \rightarrow X', L_1, \dots, L_r]$ to $[pr_2: Y \times_{X'} X \rightarrow X, pr_1^*(L_1), \dots, pr_1^*(L_r)]$.

(3) Let L be a line bundle on a scheme X . The *first Chern class operator of L* is defined to be the homomorphism $\tilde{c}_1(L): \mathcal{Z}_*(X) \rightarrow \mathcal{Z}_{*-1}(X)$ given by $[f: Y \rightarrow X, L_1, \dots, L_r] \mapsto [f: Y \rightarrow X, L_1, \dots, L_r, f^*(L)]$. If X is smooth, we define the *first Chern class $c_1(L)$* of L to be the cobordism cycle $c_1(L) := [\text{Id}_X: X \rightarrow X, L]$.

(4) For $X, Y \in \mathbf{Sch}_k$, we define the *external product*

$$\times: \mathcal{Z}_*(X) \times \mathcal{Z}_*(Y) \rightarrow \mathcal{Z}_*(X \times Y)$$

by sending the pair $[f: X' \rightarrow X, L_1, \dots, L_r] \times [g: Y' \rightarrow Y, M_1, \dots, M_s]$ to

$$[f \times g: X' \times Y' \rightarrow X \times Y, pr_1^*(L_1), \dots, pr_1^*(L_r), pr_2^*(M_1), \dots, pr_2^*(M_s)].$$

The functor $\mathcal{Z}_*(-)$ defines the universal oriented Borel-Moore functor on \mathbf{Sch}_k with products in the sense of [12, Definition 2.1.10]. This universality is based on the observation in [ibid., Remark 2.1.8] that, in $\mathcal{Z}_*(X)$ we have the equality $[f: Y \rightarrow X, L_1, \dots, L_r] = f_* \circ \tilde{c}_1(L_r) \circ \dots \circ \tilde{c}_r(L_1) \circ \pi_Y^*(1)$, where $\pi_Y: Y \rightarrow \text{Spec}(k)$ is the structure map and $1 := 1_{\text{Spec}(k)} \in \mathcal{Z}_0(k)$.

2.2. Algebraic equivalence on vector bundles. For algebraic cycles on schemes, the notion of algebraic equivalence was defined first in [17]. For $X \in \mathbf{Sch}_k$, we say that two algebraic cycles Z_1 and Z_2 on X are *algebraically equivalent*, if there exists a smooth projective connected curve C and k -rational points t_1, t_2 on C and a cycle Z on $X \times C$ such that $Z|_{X \times \{t_j\}} = Z_j$ for $j = 1, 2$. We refer to [6, Chapter 10] for basic facts on algebraic equivalence of algebraic cycles. For vector bundles, we have a related notion. We say two vector bundles E_1, E_2 of finite rank on X are *algebraically equivalent*, if there is a smooth projective connected curve C , k -rational points t_1, t_2 on C , and a vector bundle V on $X \times C$ such that $E_i \simeq V|_{X \times \{t_j\}}$ for $j = 1, 2$. We use \sim_{alg} to mean both of the above notions on cycles and vector bundles.

We say that a vector bundle E of rank m on X is *algebraically trivial* if it is algebraically equivalent to the trivial bundle $O_X^{\oplus m}$. The following facts about algebraic equivalence of vector bundles and algebraic cycles will be useful in the sequel.

Lemma 2.3. *Two vector bundles E_1 and E_2 on a scheme X are algebraically equivalent if and only if $E_1 \otimes L$ and $E_2 \otimes L$ are algebraically equivalent for every $L \in \text{Pic}(X)$.*

Lemma 2.4. *Let X be a smooth scheme and let D_1 and D_2 be two Weil divisors on X . Then $D_1 \sim_{\text{alg}} D_2$ as cycles if and only if $O_X(D_1) \sim_{\text{alg}} O_X(D_2)$ as line bundles.*

Proof. If D_1 and D_2 are algebraically equivalent, then there is a smooth connected scheme T of dimension > 0 , k -rational points t_1, t_2 on T , and a Weil divisor D on $X \times T$ such that $D_1 - D_2 = D_{t_1} - D_{t_2}$. We can assume that T is projective.

By [10, Theorem 1] (see also [6, Example 10.3.2] if k is algebraically closed), we can replace T by a smooth projective curve C passing through t_1, t_2 . Thus, we have a Weil divisor D on $X \times C$ such that $D_1 - D_2 = D_{t_1} - D_{t_2}$. We can modify D by $D - (D_{t_2} \times C) + (D_{t_1} \times C)$ so that $D_{t_i} = D_i$ for $i = 1, 2$. Letting $\mathcal{L} = O_{X \times C}(D)$, we see that $\mathcal{L}|_{X \times \{t_i\}} \simeq O_X(D_i)$ for $i = 1, 2$.

Conversely, suppose there is a line bundle \mathcal{L} on $X \times C$ such that $\mathcal{L}|_{X \times \{t_i\}} \simeq O_X(D_i)$ for $i = 1, 2$. Since $X \times C$ is smooth, there is a Weil divisor D on $X \times C$ whose associated

line bundle is \mathcal{L} . This implies in particular that $D_{t_i} \sim_{\text{rat}} D_i$ for $i = 1, 2$. In other words, we have $D_1 \sim_{\text{rat}} D_{t_1} \sim_{\text{alg}} D_{t_2} \sim_{\text{rat}} D_2$, which implies that $D_1 \sim_{\text{alg}} D_2$. \square

Remark 2.5. Note that if the curve C in the above definition is (a nonempty open subset of) \mathbb{P}^1 , then we can say that E_1 and E_2 are rationally equivalent. However, when X is semi-normal, this is equivalent to saying that E_1 and E_2 are isomorphic.

3. THE ALGEBRAIC COBORDISM Ω_*^{alg} MODULO ALGEBRAIC EQUIVALENCE

In this section, we define an algebraic cobordism theory of a scheme X associated to algebraic equivalence. The starting point is the simple observation that the algebraic cobordism $\Omega_*(X)$ is associated to the rational equivalence of line bundles in that, two line bundles on a smooth scheme are isomorphic if and only if they are rationally equivalent (see Remark 2.5).

Levine and Morel constructed $\Omega_*(X)$ from the cobordism cycles $\mathcal{Z}_*(X)$ of Definition 2.1. We will define the cobordism theory $\Omega_*^{\text{alg}}(X)$ that is similar to that of Levine-Morel, with one additional set of relations that identifies two integral cobordism cycles when their line bundles are suitably related by algebraic equivalence. Recall our notation of using \sim for algebraic equivalence.

Definition 3.1 (Compare with [12, Definition 2.4.5]). For $X \in \mathbf{Sch}_k$, the \sim -pre-cobordism $\underline{\Omega}_*^{\text{alg}}(X)$ is the quotient of $\mathcal{Z}_*(X)$ by the following three relations:

- (1) (Dim) If there is a smooth quasi-projective morphism $\pi: Y \rightarrow Z$ with line bundles $M_1, \dots, M_{s > \dim Z}$ on Z with $L_i \simeq \pi^* M_i$ for $i = 1, \dots, s \leq r$, then $[f: Y \rightarrow X, L_1, \dots, L_r] = 0$.
- (2) (Sect) For a section $s: Y \rightarrow L$ of a line bundle L on Y with its smooth associated divisor $i: D \rightarrow Y$, we impose

$$[f: Y \rightarrow X, L_1, \dots, L_r, L] = [f \circ i: D \rightarrow X, i^* L_1, \dots, i^* L_r].$$

- (3) (Equiv) $[Y \rightarrow X, L_1, \dots, L_r]$ and $[Y' \rightarrow X, L'_1, \dots, L'_r]$ are identified if there exists an isomorphism $\phi: Y \rightarrow Y'$ over X , a permutation σ of $\{1, \dots, r\}$ and algebraic equivalences of the line bundles $L_i \sim \phi^*(L'_{\sigma(i)})$.

It is immediate from the definition that there is a natural surjection $\Phi_X: \Omega_*(X) \rightarrow \underline{\Omega}_*^{\text{alg}}(X)$.

Remark 3.2. If we take the quotient of $\mathcal{Z}_*(X)$ by only the relations (Dim) and (Sect), then the resulting quotient group is the pre-cobordism $\underline{\Omega}_*(X)$ of Levine-Morel in *ibid*. The cobordism cycles of the form $[Y \rightarrow X, L] - [Y \rightarrow X, L']$ are zero in $\underline{\Omega}_*(X)$ if $L \simeq L'$. If \sim in (Equiv) is replaced by the rational equivalence \sim_{rat} of line bundles, then by Remark 2.5, the modified relation (Equiv)_{rat} plays no role because Y is smooth, thus semi-normal.

Lemma 3.3. *All four operations (projective push-forward, smooth pull-back, external product and the first Chern class operation) in Definition 2.2 descend onto the \sim -pre-cobordism $\underline{\Omega}_*^{\text{alg}}$.*

Proof. By [12, Remarks 2.1.11, 2.1.14, Lemmas 2.4.2, 2.4.7], $\underline{\Omega}_*$ is an oriented Borel-Moore functor on \mathbf{Sch}_k with product in the sense of [12, Definition 2.1.10]. This implies that (Dim) and (Sect) are respected by the four operations. For (Equiv), it follows from the fact that the pull-back operations on line bundles via any morphisms respect algebraic equivalence. \square

To impose the formal group law into our cobordism theory as in [12, p. 4, §2.4.4], first recall from *ibids.* that there is a graded polynomial ring $\mathbb{Z}[a_{i,j} | i, j \geq 0]$, where $a_{i,j}$ are variables of degree $i + j - 1$ subject to some relations. This is called the Lazard ring, written \mathbb{L}_* . There is a power series $F_{\mathbb{L}_*}(u, v) := \sum_{i,j} a_{i,j} u^i v^j \in \mathbb{L}_*[[u, v]]$ such that the pair $(\mathbb{L}_*, F_{\mathbb{L}_*})$ is the universal commutative formal group law of rank one. One also uses the cohomological indexing \mathbb{L}^* by letting $\mathbb{L}^n = \mathbb{L}_{-n}$. We have $\mathbb{L}^0 \simeq \mathbb{Z}$ and $\mathbb{L}^{-n} = \mathbb{L}_n = 0$ if $n < 0$. Now we define the main object of study of this paper.

Definition 3.4 (Compare with [12, Definition 2.4.10]). For $X \in \mathbf{Sch}_k$, the graded group $\Omega_*^{\text{alg}}(X)$ is defined to be the quotient of $\mathbb{L}_* \otimes_{\mathbb{Z}} \underline{\Omega}_*^{\text{alg}}(X)$ by the relations (FGL) of the form $F_{\mathbb{L}_*}(\tilde{c}_1(L), \tilde{c}_1(M))([f: Y \rightarrow X, L_1, \dots, L_r]) = \tilde{c}_1(L \otimes M)([f: Y \rightarrow X, L_1, \dots, L_r])$ for line bundles L and M on X . By the relation (Dim) in Definition 3.1-(1), the expression $F_{\mathbb{L}_*}(\tilde{c}_1(L), \tilde{c}_1(M))$ is a finite sum so that the operator is well-defined. This graded abelian group $\Omega_*^{\text{alg}}(X)$ is called the *algebraic cobordism of X modulo algebraic equivalence*.

When X is smooth and equidimensional of dimension n , the codimension of a cobordism d -cycle is defined to be $n - d$. We set $\Omega_d^{\text{alg}}(X) := \Omega_d^{\text{alg}}(X)$, and $\Omega_{\text{alg}}^*(X)$ is the direct sum of the groups over the all codimensions.

If we omit the relation (Equiv) in the above process, we obtain the algebraic cobordism theory $\Omega_*(X)$ of [12, Definition 2.4.10]. In particular, we have a natural surjection $\Phi_X: \Omega_*(X) \rightarrow \Omega_*^{\text{alg}}(X)$. We immediately see the following:

Proposition 3.5. *All four operations (projective push-forward, smooth pull-back, exterior product, and the first Chern class operation) in Definition 2.2 descend onto the cobordism Ω_*^{alg} .*

Remark 3.6. By definition, we have a natural ring homomorphism

$$(3.1) \quad \Phi^{\text{alg}}: \mathbb{L}_* \rightarrow \Omega_*^{\text{alg}}(k)$$

induced from the quotient map $\mathbb{L}_* \otimes_{\mathbb{Z}} \underline{\Omega}_*^{\text{alg}}(k) \rightarrow \Omega_*^{\text{alg}}(k)$, which factors through the known map $\Phi: \mathbb{L}_* \rightarrow \Omega_*(k)$ in [12, p.39]. We will see later in Proposition 9.2 that this is an isomorphism.

We have a natural map $q^{\text{alg}}: \underline{\Omega}_*^{\text{alg}}(X) \rightarrow \Omega_*^{\text{alg}}(X)$. It was proven in [12, Lemma 2.5.9] that the map $q: \underline{\Omega}_*(X) \rightarrow \Omega_*(X)$ is surjective. We have a similar result:

Lemma 3.7. *Given any scheme X , the abelian group $\Omega_*^{\text{alg}}(X)$ is generated by the images of the integral cobordism cycles $[Y \rightarrow X, L_1, \dots, L_r]$. In other words, the natural map $\underline{\Omega}_*(X) \rightarrow \Omega_*^{\text{alg}}(X)$ is surjective.*

Proof. It suffices to show that the map $q^{\text{alg}}: \underline{\Omega}_*^{\text{alg}} \rightarrow \Omega_*^{\text{alg}}(X)$ is surjective. But, this follows from the observation that in the commutative diagram

$$(3.2) \quad \begin{array}{ccc} \underline{\Omega}_*(X) & \xrightarrow{\Phi_X} & \underline{\Omega}_*^{\text{alg}}(X) \\ q \downarrow & & q^{\text{alg}} \downarrow \\ \Omega_*(X) & \xrightarrow{\Phi_X} & \Omega_*^{\text{alg}}(X), \end{array}$$

the map Φ_X is clearly surjective and q is surjective by [12, Lemma 2.5.9]. \square

Our discussion so far summarizes as follows (compare with [12, Theorem 2.4.13]):

Proposition 3.8. *The theory Ω_*^{alg} is an oriented Borel-Moore \mathbb{L}_* -functor on \mathbf{Sch}_k of geometric type in the sense of [12, Definitions 2.1.2, 2.1.12, 2.2.1].*

In the rest of this section, we shall prove the following universal property of Ω_*^{alg} .

Definition 3.9. Let A_* be an oriented Borel-Moore \mathbb{L}_* -functor on \mathbf{Sch}_k of geometric type. We say that A_* *respects algebraic equivalence*, if for any $X \in \mathbf{Sch}_k$ and for any pair of algebraically equivalent line bundles L and M over X , we have $\tilde{c}_1(L) = \tilde{c}_1(M)$ as operators $A_*(X) \rightarrow A_{*-1}(X)$.

We say that an oriented cohomology theory A^* on \mathbf{Sm}_k *respects algebraic equivalence*, if for $X \in \mathbf{Sm}_k$ and a pair of algebraically equivalent line bundles L and M over X , we have $\tilde{c}_1(L) = \tilde{c}_1(M)$ as operators $A^*(X) \rightarrow A^{*+1}(X)$.

Proposition 3.10. *The theory Ω_*^{alg} is universal among all oriented Borel-Moore \mathbb{L}_* -functors on \mathbf{Sch}_k of geometric type that respect algebraic equivalence. In other words, for any theory A_* satisfying Definition 3.9, there exists a unique morphism $\theta_A: \Omega_*^{\text{alg}} \rightarrow A_*$ of oriented Borel-Moore \mathbb{L}_* -functor of geometric type on \mathbf{Sch}_k .*

We shall prove this proposition following a series of deductions. These intermediate results provide useful information on the relationship between Ω_* and Ω_*^{alg} .

Lemma 3.11. *The kernel of the map $\Phi_X: \underline{\Omega}_*(X) \rightarrow \underline{\Omega}_*^{\text{alg}}(X)$ is a subgroup generated by elements of the form $[f: Y \rightarrow X, L] - [f: Y \rightarrow X, M]$ with $L \sim M$.*

Proof. Let $\theta_X: \underline{\Omega}_*(X) \twoheadrightarrow \underline{\Omega}_*^{\text{alg}}(X)$ be the quotient of $\underline{\Omega}_*(X)$ by the subgroup generated by elements given in the lemma. It follows from the definition and the surjection $\mathcal{Z}_*(X) \twoheadrightarrow \underline{\Omega}_*(X)$ that $\ker(\Phi_X)$ is generated by elements of the form $\eta = [f: Y \rightarrow X, L_1, \dots, L_r] - [f': Y' \rightarrow X, L'_1, \dots, L'_r]$, where $\phi: Y \rightarrow Y'$ is an isomorphism over X and σ is a permutation of $\{1, \dots, r\}$ such that $L_i \sim \phi^*(L'_{\sigma(i)})$. It suffices to show that such elements vanish in $\underline{\Omega}_*^{\text{alg}}(X)$. We can modify η so that $\eta = [f: Y \rightarrow X, L_1, \dots, L_r] - [f: Y \rightarrow X, L'_1, \dots, L'_r]$, where $L_i \sim L'_i$ for $1 \leq i \leq r$ by virtue of the relations in $\underline{\Omega}_*(X)$ as described in Definition 2.1. Since $\theta_X(\eta) = f_* \circ \theta_Y\{\tilde{c}_1(L_1) \circ \dots \circ \tilde{c}_1(L_r)(1_Y) - \tilde{c}_1(L'_1) \circ \dots \circ \tilde{c}_1(L'_r)(1_Y)\}$, it is enough to consider the case when $X = Y$ and $f = \text{Id}_Y$. The lemma now follows by repeated applications of the Chern class operators, i.e., $\tilde{c}_1(L_1) \circ \dots \circ \tilde{c}_1(L_r)(1_Y) = \tilde{c}_1(L'_1) \circ \dots \circ \tilde{c}_1(L'_r)(1_Y)$ in $\underline{\Omega}_*^{\text{alg}}(Y)$. \square

For $X \in \mathbf{Sch}_k$, let $\tilde{\mathcal{R}}_*^{\text{alg}}(X)$ denote (compare with [12, Definition 2.5.13]) the graded subgroup of $\underline{\Omega}_*^{\text{alg}}(X)$ generated by elements of the form

$$(3.3) \quad f_* \circ \tilde{c}_1(L_1) \circ \dots \circ \tilde{c}_1(L_r) \{F(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) - \tilde{c}_1(L \otimes M)(\eta)\},$$

where $[f: Y \rightarrow X, L_1, \dots, L_r]$ is a standard cobordism cycle, $L, M \in \text{Pic}(Y)$ and $\eta \in \underline{\Omega}_*^{\text{alg}}(Y)$. Since we have a natural surjection $\mathcal{Z}_*(k) \twoheadrightarrow \Omega_*(k)$ (see [12, Lemma 2.5.9]) and the isomorphism $\Phi: \mathbb{L}_* \xrightarrow{\sim} \Omega_*(k)$ (see [12, Theorem 1.2.7]), each element $a_{i,j} \in \mathbb{L}_*$ has a lift in $\mathcal{Z}_*(k)$. In particular, the elements of the form $F(\tilde{c}_1(L), \tilde{c}_1(M))(\eta)$ are well-defined in $\underline{\Omega}_*(Y)$, thus well-defined in $\underline{\Omega}_*^{\text{alg}}(Y)$. Set $\tilde{\Omega}_*^{\text{alg}}(X) := \underline{\Omega}_*^{\text{alg}}(X) / \tilde{\mathcal{R}}_*^{\text{alg}}(X)$. The following result is a refinement of Lemma 3.7.

Proposition 3.12. *For any $X \in \mathbf{Sch}_k$, there is a natural map $\psi_X^{\text{alg}}: \tilde{\Omega}_*^{\text{alg}}(X) \rightarrow \Omega_*^{\text{alg}}(X)$ which is an isomorphism.*

Proof. It follows from Definition 3.4 that the map $\underline{\Omega}_*^{\text{alg}}(X) \rightarrow \Omega_*^{\text{alg}}(X)$ kills $\tilde{\mathcal{R}}_*^{\text{alg}}(X)$. This induces the natural map $\psi_X^{\text{alg}}: \tilde{\Omega}_*^{\text{alg}}(X) \rightarrow \Omega_*^{\text{alg}}(X)$. We have already shown in Lemma 3.7 that this map is surjective. We define an inverse $\phi_X^{\text{alg}}: \Omega_*^{\text{alg}}(X) \rightarrow \tilde{\Omega}_*^{\text{alg}}(X)$ of ψ_X^{alg} to complete the proof of the proposition.

To do this, we consider the commutative diagram

$$(3.4) \quad \begin{array}{ccccc} \underline{\Omega}_*(X) & \xrightarrow{\quad} & \underline{\Omega}_*^{\text{alg}}(X) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & \mathbb{L}_* \otimes \underline{\Omega}_*(X) & \xrightarrow{\quad \bar{\beta} \quad} & \mathbb{L}_* \otimes \underline{\Omega}_*^{\text{alg}}(X) & \\ & \downarrow \alpha & & \downarrow \alpha^{\text{alg}} & \\ \tilde{\Omega}_*(X) & \xrightarrow{\quad} & \tilde{\Omega}_*^{\text{alg}}(X) & & \\ \swarrow \psi_X & & \swarrow \psi_X^{\text{alg}} & & \\ & \Omega_*(X) & \xrightarrow{\quad \beta \quad} & \Omega_*^{\text{alg}}(X) & \end{array}$$

(Note: The diagram also includes dashed arrows: \$\gamma\$ from \$\mathbb{L}_* \otimes \underline{\Omega}_*(X)\$ to \$\tilde{\Omega}_*(X)\$, \$\gamma^{\text{alg}}\$ from \$\mathbb{L}_* \otimes \underline{\Omega}_*^{\text{alg}}(X)\$ to \$\tilde{\Omega}_*^{\text{alg}}(X)\$, \$\phi_X\$ from \$\tilde{\Omega}_*(X)\$ to \$\Omega_*(X)\$, and \$\phi_X^{\text{alg}}\$ from \$\tilde{\Omega}_*^{\text{alg}}(X)\$ to \$\Omega_*^{\text{alg}}(X)\$.)

where $\tilde{\Omega}_*(X)$ is defined in [12, Definition 2.5.13]. All the squares in the above diagram commute and the maps ψ_X and ϕ_X are inverses of each other by [12, Proposition 2.5.15]. By Lemma 3.11, the kernel of the map $\bar{\beta}$ is generated by elements of the form $a \otimes ([Y \rightarrow X, L] - [Y \rightarrow X, M])$, where $L \sim M$ and $a \in \mathbb{L}_*$. On the other hand, such an element maps to $\Phi(a) ([Y \rightarrow X, L] - [Y \rightarrow X, M])$ in $\tilde{\Omega}_*(X)$ under $\phi_X \circ \alpha$ (see (3.1)). In particular, these elements are killed in $\tilde{\Omega}_*^{\text{alg}}(X)$ under the composite map $\gamma: \mathbb{L}_* \otimes \underline{\Omega}_*(X) \rightarrow \Omega_*(X) \rightarrow \tilde{\Omega}_*(X) \rightarrow \tilde{\Omega}_*^{\text{alg}}(X)$. Thus it descends to the quotient $\gamma^{\text{alg}}: \mathbb{L}_* \otimes \underline{\Omega}_*^{\text{alg}}(X) \rightarrow \tilde{\Omega}_*^{\text{alg}}(X)$.

Next, we see from Definition 3.4 that the kernel of α^{alg} is generated by elements of the form $F_{\mathbb{L}_*}(\tilde{c}_1(L), \tilde{c}_1(M))([f: Y \rightarrow X, L_1, \dots, L_r]) - \tilde{c}_1(L \otimes M)([f: Y \rightarrow X, L_1, \dots, L_r])$ for line bundles L_i on Y , and line bundles L and M on X . But these elements also lie in the kernel of the map α . In particular, they die in $\tilde{\Omega}_*^{\text{alg}}(X)$ via γ from which we conclude that $\ker(\alpha^{\text{alg}}) \subseteq \ker(\gamma^{\text{alg}})$. Hence, the map γ^{alg} descends to $\phi_X^{\text{alg}}: \Omega_*^{\text{alg}}(X) \rightarrow \tilde{\Omega}_*^{\text{alg}}(X)$ which makes all the squares commute. It is clear from the construction that $\phi_X^{\text{alg}} \circ \psi_X^{\text{alg}}$ is the identity map. In particular, ψ_X^{alg} is injective, thus an isomorphism. \square

Proposition 3.13. *For $X \in \mathbf{Sch}_k$, the kernel of the natural surjection $\Phi_X: \Omega_*(X) \rightarrow \Omega_*^{\text{alg}}(X)$ is the graded subgroup generated by the cobordism cycles of the form $[f: Y \rightarrow X, L] - [f: Y \rightarrow X, M]$, where L and M are algebraically equivalent.*

Proof. In the commutative diagram

$$(3.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \tilde{\mathcal{R}}_*(X) & \longrightarrow & \underline{\Omega}_*(X) & \longrightarrow & \Omega_*(X) \longrightarrow 0 \\ & & \downarrow & & \downarrow \Phi_X & & \downarrow \Phi_X \\ 0 & \longrightarrow & \tilde{\mathcal{R}}_*^{\text{alg}}(X) & \longrightarrow & \underline{\Omega}_*^{\text{alg}}(X) & \longrightarrow & \Omega_*^{\text{alg}}(X) \longrightarrow 0, \end{array}$$

the top row is exact by [12, Proposition 2.5.15] and the bottom row is exact by Proposition 3.12. The left vertical arrow in this diagram is surjective by the definition of $\tilde{\mathcal{R}}_*^{\text{alg}}(X)$ above and that of $\tilde{\mathcal{R}}_*(X)$ in [12, Lemma 2.5.14]. Hence, the map $\ker(\Phi_X) \rightarrow \ker(\Phi_X)$ is surjective by the snake lemma. On the other hand, Lemmas 2.3 and 3.11 imply that the group $\ker(\Phi_X)$ is generated by cobordism cycles of the form $[f: Y \rightarrow X, L] - [f: Y \rightarrow X, M]$ where $L \sim M$. This proves the proposition. \square

Proof of Proposition 3.10: The theory Ω_*^{alg} satisfies Definition 3.9 in view of the relation (Equiv) of Definition 3.1. To prove its universality, we first recall from [12, Theorem 2.4.13] that the algebraic cobordism Ω_* of Levine-Morel is a universal oriented

Borel-Moore \mathbb{L}_* -functor of geometric type. So, there is a morphism $\theta: \Omega_* \rightarrow A_*$ of oriented Borel-Moore \mathbb{L}_* -functor of geometric type on \mathbf{Sch}_k .

To show that it induces $\theta_A: \Omega_*^{\text{alg}} \rightarrow A_*$, it is enough to show using Proposition 3.13 that $\theta(\eta) = 0$ in $A_*(X)$ for $\eta = [f: Y \rightarrow X, L] - [f: Y \rightarrow X, M]$, where $L \sim M$. This is equivalent to $f_*((\tilde{c}_1(L) - \tilde{c}_1(M))(1_Y)) = 0 \in A_*(X)$. But this holds by the assumption that $\tilde{c}_1(L) = \tilde{c}_1(M)$ on $A_*(Y)$. Hence, we have the induced morphism $\theta_A: \Omega_*^{\text{alg}} \rightarrow A_*$ as desired. \square

Some fundamental properties of Ω_*^{alg} will be studied in § 6.2 and § 7. We shall also show that Ω_*^{alg} is an oriented cohomology theory on \mathbf{Sm}_k and an oriented Borel-Moore homology theory on \mathbf{Sch}_k (see [12, Definitions 1.1.2, 5.1.3]), equipped with a similar universal property.

4. ALGEBRAIC DOUBLE-POINT COBORDISM ω_*^{alg}

In this section, we recall the cobordism theory ω_* of [13] based on the double-point relations and study its *algebraic equivalence* analogue ω_*^{alg} following the suggestion of Levine and Pandharipande in [ibid., §11.2].

4.1. Double-point cobordism after Levine-Pandharipande. The description of ω_* by Levine and Pandharipande is simpler than that of Levine and Morel's Ω_* in [12] in the following respects: first, the cobordism cycles are simpler, *i.e.*, without the attached line bundles and the artificial imposition of the formal group law as in Definitions 2.1 and 3.4, and second, the relations are given by a single sort of morphisms called *double-point degenerations*.

The cobordism cycles in the sense of Levine-Pandharipande, recalled below, will also be called cobordism cycles whenever no confusion arises.

Definition 4.1 ([13, §0.2]). Let $X \in \mathbf{Sch}_k$. An *integral cobordism cycle* on X is the isomorphism class over X of a projective morphism $f: Y \rightarrow X$, where Y is smooth and integral. This will be denoted by $[f: Y \rightarrow X]$. Its dimension is by definition $\dim Y$. If $Y = \coprod Y_i \in \mathbf{Sm}_k$ where each Y_i is integral, then given a projective morphism $f: Y \rightarrow X$, the cobordism cycle $[f: Y \rightarrow X]$ is defined to be the sum of $[f|_{Y_i}: Y_i \rightarrow X]$. Let $\mathcal{M}_*(X)^+$ be the free abelian group on the set of all integral cobordism cycles over X , and let $\mathcal{M}_d(X)^+$ be its subgroup generated by the cobordism cycles of dimension d . An element of $\mathcal{M}_*(X)^+$ will be called a *cobordism cycle*.

Now we recall the notion of double-point degenerations and the associated relations from [13, §0.2, §0.3, §11.2].

Definition 4.2. Let $Y \in \mathbf{Sm}_k$ be of pure dimension. Let (C, p) be a pair consisting of a smooth projective connected curve C and a k -rational point $p \in C$.

(1) A morphism $\pi: Y \rightarrow C$ of scheme is a *double-point degeneration* over $p \in C$ if $\pi^{-1}(p)$ can be written as $\pi^{-1}(p) = A \cup B$, where A and B are smooth closed subschemes of Y of codimension 1 intersecting transversally. The intersection $D = A \cap B$ is called the *double-point locus* of π over $p \in C$. We allow A, B , or D to be empty. Let $N_{A/D}$ and $N_{B/D}$ denote the normal bundles of D in A and B , respectively. As in [13, §0.2], the projective bundles $\mathbb{P}(O_D \oplus N_{A/D})$ and $\mathbb{P}(O_D \oplus N_{B/D})$ over D are isomorphic. Either of these is denoted by $\mathbb{P}(\pi) \rightarrow D$.

(2) Let $X \in \mathbf{Sch}_k$ and let pr_1, pr_2 be the projections from $X \times C$ to X and C , respectively. Let $Y \in \mathbf{Sm}_k$ be of pure dimension, and let $g: Y \rightarrow X \times C$ be a projective morphism such that $\pi = pr_2 \circ g: Y \rightarrow C$ is a double-point degeneration over $p \in C$. For each regular value $\zeta \in C(k)$ of π , the triple (g, p, ζ) is called a *double-point cobordism*.

with the degenerate fiber over $p \in C$ and the smooth fiber over ζ . The associated *double-point relation* over X is given by

$$\partial_C(g, p, \zeta) := [Y_\zeta \rightarrow X] - [A \rightarrow X] - [B \rightarrow X] + [\mathbb{P}(\pi) \rightarrow X] = 0 \text{ in } \mathcal{M}_*(X)^+,$$

where $Y_\zeta := \pi^{-1}(\zeta)$.

(3) Let $\mathcal{R}_*^{\text{rat}}(X) \subset \mathcal{M}_*(X)^+$ be the subgroup generated by all double-point relations over X over the pair $(C, p) = (\mathbb{P}^1, 0)$. This is the group of *rational double-point relations*. This group was denoted by $\mathcal{R}_*(X)$ in [13].

(4) Let $\mathcal{R}_*^{\text{alg}}(X) \subset \mathcal{M}_*(X)^+$ be the subgroup generated by all double-point relations over X over all pairs (C, p) of smooth projective connected curve C and a point $p \in C(k)$. This is the group of *algebraic double-point relations*.

Definition 4.3 (Levine-Pandharipande). Let $X \in \mathbf{Sch}_k$.

(1) The *(rational) double-point cobordism theory* $\omega_*(X)$ is the quotient

$$\omega_*(X) = \mathcal{M}_*(X)^+ / \mathcal{R}_*^{\text{rat}}(X).$$

(2) The *algebraic double-point cobordism theory* $\omega_*^{\text{alg}}(X)$ is the quotient

$$\omega_*^{\text{alg}}(X) = \mathcal{M}_*(X)^+ / \mathcal{R}_*^{\text{alg}}(X).$$

4.2. Basic structures. Some of the following basic properties of ω_*^{alg} follow essentially from the definition and some analogous constructions in [13, §3.1].

Proposition 4.4. *The functor $X \mapsto \omega_*^{\text{alg}}(X)$ on \mathbf{Sch}_k has the following structures.*

(1) Projective push-forward: *For a projective morphism $g: X \rightarrow X'$, we have $g_*: \omega_*^{\text{alg}}(X) \rightarrow \omega_*^{\text{alg}}(X')$ given by $g_*([f: Y \rightarrow X]) = [g \circ f: Y \rightarrow X']$. This satisfies $(g_1 \circ g_2)_* = g_{1*} \circ g_{2*}$ when g_1 and g_2 are both projective.*

(2) Smooth pull-back: *For a smooth quasi-projective morphism $g: X' \rightarrow X$ of relative dimension d , we have $g^*: \omega_*^{\text{alg}}(X) \rightarrow \omega_{*+d}^{\text{alg}}(X')$ given by $g^*([f: Y \rightarrow X]) = [pr_2: Y \times_X X' \rightarrow X']$. This satisfies $(g_1 \circ g_2)^* = g_2^* \circ g_1^*$ when g_1 and g_2 are both smooth.*

(3) External product: *We have $\times: \omega_*^{\text{alg}}(X) \times \omega_*^{\text{alg}}(X') \rightarrow \omega_*^{\text{alg}}(X \times X')$ given by $[f: Y \rightarrow X] \times [f': Y' \rightarrow X'] = [f \times f': Y \times Y' \rightarrow X \times X']$.*

(4) Unit: *The class $1_{\text{Spec}(k)} \in \omega_0^{\text{alg}}(k)$ is the unit for the external product on ω_*^{alg} .*

(5) Chern classes: *For every line bundle L on X , there is a Chern class operation $\tilde{c}_1(L): \omega_*^{\text{alg}}(X) \rightarrow \omega_{*-1}^{\text{alg}}(X)$ which is compatible with smooth pull-back and projective push-forward.*

Proof. (1) Given a projective morphism $g: X \rightarrow X'$, we already have $g_*: \mathcal{M}_*(X)^+ \rightarrow \mathcal{M}_*(X')^+$. It remains to show that g_* sends the algebraic double-point relations $\mathcal{R}_*^{\text{alg}}(X)$ into $\mathcal{R}_*^{\text{alg}}(X')$. Indeed, given an algebraic double-point cobordism (h, p, ζ) over X , where $h: Y \rightarrow X \times C$ with a smooth projective connected curve C , we get an algebraic double-point cobordism $((g \times \text{Id}_C) \circ h, p, \zeta)$ over X' , where $(g \times \text{Id}_C) \circ h: Y \rightarrow X' \times C$. We immediately note that $g_*(\partial_C(h, p, \zeta)) = \partial_C((g \times \text{Id}_C) \circ h, p, \zeta)$. This proves (1).

(2) Given a smooth and quasi-projective morphism $g: X' \rightarrow X$, we have $g^*: \mathcal{M}_*(X)^+ \rightarrow \mathcal{M}_*(X')^+$. It remains to show that g^* sends $\mathcal{R}_*^{\text{alg}}(X)$ into $\mathcal{R}_*^{\text{alg}}(X')$. This follows by observing that given an algebraic double-point cobordism (h, p, ζ) over X , the pull-back (g^*h, p, ζ) , given by the second projection of the fiber product $Y': = Y \times_{X \times C} (X' \times C) \rightarrow X' \times C$, is an algebraic double-point cobordism over X' .

(3) The map $\times: \mathcal{M}_*(X)^+ \times \mathcal{M}_*(X')^+ \rightarrow \mathcal{M}_*(X \times X')^+$ is defined on the level of cobordism cycles. For an algebraic double-point cobordism (h, p, ζ) over X as before, for each $[f: Y' \rightarrow X'] \in \mathcal{M}_*(X')^+$, we get an induced algebraic double-point cobordism

$(h \times f, p, \zeta)$, where $h \times f: Y \times Y' \rightarrow X \times X' \times C$. Similarly, interchanging the role of X and X' , we see that \times descends onto the level of $\omega_*^{\text{alg}}(-)$. This proves (3). Part (4) is immediate.

(5) The construction of the first Chern class operation $\tilde{c}_1(L)$ on $\omega_*^{\text{alg}}(X)$ follows the same arguments as for $\omega_*(X)$ by first assuming that L is globally generated and then deducing the general case, as in [13, §4 and §9]. We omit the details. \square

5. THE BASIC EXACT SEQUENCE

Let $X \in \mathbf{Sch}_k$. By [13, Theorem 1], the natural map $\omega_*(X) \rightarrow \Omega_*(X)$ is an isomorphism. We often use this identification implicitly. Let (C, t_1, t_2) denote a smooth projective connected curve C with distinct points $t_1, t_2 \in C(k)$ with the inclusions $i_j: X \times \{t_j\} \rightarrow X \times C$. By the existence of the l.c.i. pull-backs on Ω_* in [12, §6.5], we have maps $i_1^*, i_2^*: \omega_*(X \times C) \rightarrow \omega_*(X)$. By definition, we also have a natural surjection $\Psi_X: \omega_*(X) \rightarrow \omega_*^{\text{alg}}(X)$. The main theorem of the section is:

Theorem 5.1. *Let $X \in \mathbf{Sch}_k$. The sequence*

$$\bigoplus_{(C, t_1, t_2)} \omega_*(X \times C) \xrightarrow{i_1^* - i_2^*} \omega_*(X) \xrightarrow{\Psi_X} \omega_*^{\text{alg}}(X) \rightarrow 0,$$

where (C, t_1, t_2) runs over the equivalence classes of all triples consisting of a smooth projective connected curve C and two distinct points $t_1, t_2 \in C(k)$, is exact.

We begin with some remarks on cobordism cycles associated to strict normal crossing divisors on smooth schemes.

5.1. Remarks on divisor classes. Recall from [12, §3.1] that given a strict normal crossing divisor E on $Y \in \mathbf{Sm}_k$ with the support $\iota: |E| \rightarrow Y$, there is a class $[E \rightarrow |E|] \in \Omega_*(|E|)$ that satisfies $\iota_*([E \rightarrow |E|]) = [Y \rightarrow Y, O_Y(E)] = \tilde{c}_1(O_Y(E))(1_Y)$. Since we have a natural surjection $\Omega_* \rightarrow \Omega_*^{\text{alg}}$, the class $[E \rightarrow |E|]$ makes sense also in $\Omega_*^{\text{alg}}(|E|)$.

The construction $[E \rightarrow |E|] \in \Omega_*(|E|)$ uses the formal group law F for $\Omega_*(k)$. We look at only the following case from [12, §3.1]. The special case we need is when $E = E_1 + E_2$, where E_1 and E_2 are transversal smooth divisors on $Y \in \mathbf{Sm}_k$. Let $\iota_D: D = E_1 \cap E_2 \rightarrow Y$ be the inclusion. We let $O_D(E_i) := \iota_D^*(O_Y(E_i))$. The class $[E \rightarrow Y] \in \Omega_*(Y)$ is defined as

$$[E \rightarrow Y] := [E_1 \rightarrow Y] + [E_2 \rightarrow Y] + \iota_{D*}(F^{1,1}(\tilde{c}_1(O_D(E_1)), \tilde{c}_1(O_D(E_2)))(1_D)),$$

where $F^{1,1}(u, v) = \sum_{i,j \geq 1} a_{i,j} u^{i-1} v^{j-1} \in \Omega_*(k)[[u, v]]$ and $a_{i,j} \in \Omega_{i+j-1}(k)$ are the coefficients of the formal group law.

In addition, suppose that $O_D(E) := \iota_D^*(O_Y(E))$ is trivial, i.e., $O_D(E_1) \simeq O_D(E_2)^{-1}$ on D . Let $\mathbb{P}_D \rightarrow D$ be the \mathbb{P}^1 -bundle $\mathbb{P}(O_D \oplus O_D(E_1))$. By [13, Lemma 3.3], we have

$$(5.1) \quad F^{1,1}(\tilde{c}_1(O_D(E_1)), \tilde{c}_1(O_D(E_2)))(1_D) = -[\mathbb{P}_D \rightarrow D] \in \Omega_*(D).$$

Hence, we have the following equation in $\Omega_*(Y)$ (and hence in $\Omega_*^{\text{alg}}(Y)$):

$$(5.2) \quad [E \rightarrow Y] - [E_1 \rightarrow Y] - [E_2 \rightarrow Y] + [\mathbb{P}_D \rightarrow Y] = 0.$$

5.2. Proof of Theorem 5.1. Consider the commutative diagram with the top exact row:

$$(5.3) \quad \begin{array}{ccccccc} 0 \rightarrow \mathcal{R}_*^{\text{alg}}(X) & \longrightarrow & \mathcal{M}_*(X)^+ & \longrightarrow & \omega_*^{\text{alg}}(X) & \longrightarrow & 0 \\ & \searrow \theta & \downarrow & & \parallel & & \\ \bigoplus_{(C, t_1, t_2)} \omega_*(X \times C) & \xrightarrow{\theta'} & \omega_*(X) & \xrightarrow{\Psi_X} & \omega_*^{\text{alg}}(X) & \longrightarrow & 0, \end{array}$$

where θ is the composition of the two arrows and θ' is the sum of the maps $i_1^* - i_2^*$. We want to prove that the bottom row is exact. It is apparent that $\ker(\Psi_X) = \text{Im}(\theta)$, thus it suffices to prove that $\text{Im}(\theta) = \text{Im}(\theta')$.

We prove $\text{Im}(\theta) \subseteq \text{Im}(\theta')$ first. Let (g, p, ζ) be a double-point cobordism as in Definition 4.2, i.e., a projective $g: Y \rightarrow X \times C$, two points $p, \zeta \in C(k)$ such that for $\pi = pr_2 \circ g$ we have $\pi^{-1}(p) = A \cup B$. Set $\gamma := [g: Y \rightarrow X \times C] \in \omega_*(X \times C)$.

Let $i_p: X \times \{p\} \rightarrow X \times C$ be the inclusion and let $X_p := X \times \{p\}$. Since the divisor $E := g^*(X_p) = A + B$ is strict normal crossing, we have $\gamma \in \Omega_*(X \times C)_{X_p}$ (see Definitions 11.1, 11.2, 11.3). By Theorem 11.4, Definition 11.5 and [12, Lemma 6.5.6], we see that $i_p^*(\gamma) = g'_*([E \rightarrow |E|]) \in \omega_*(X_p)$, where $g' = g|_{|E|}: |E| \rightarrow X_p$. Consider now the commutative diagram:

$$\begin{array}{ccccc} |E| & \xrightarrow{\iota_E} & Y & & \\ g' \downarrow & & \downarrow g & \searrow \pi' & \\ X_p & \xrightarrow{i_p} & X \times C & \xrightarrow{pr_1} & X. \end{array}$$

Note that $pr_1 \circ i_p = \text{Id}_X$ via $X \simeq X_p$ and π' is projective. Thus, $i_p^*(\gamma) = g'_*([E \rightarrow |E|]) = pr_{1*} i_{p*} g'_*([E \rightarrow |E|]) = \pi'_* \iota_{E*}([E \rightarrow |E|]) = \pi'_*([E \rightarrow Y]) = \dagger [A \rightarrow X] + [B \rightarrow X] - [\mathbb{P}(\pi) \rightarrow X]$ in $\omega_*(X)$, where \dagger follows from (5.2). Since Y_ζ is smooth, $i_\zeta^*(\alpha) = [Y_\zeta \rightarrow X]$. Hence, we get $\theta(\partial_C(g, p, \zeta)) = [Y_\zeta \rightarrow X] - [A \rightarrow X] - [B \rightarrow X] + [\mathbb{P}(\pi) \rightarrow X] = -(i_p^* - i_\zeta^*)(\gamma)$. That is, $\text{Im}(\theta) \subseteq \text{Im}(\theta')$.

To prove the reverse inclusion $\text{Im}(\theta) \supseteq \text{Im}(\theta')$, we consider two cases.

Case 1: First assume that X is smooth. For (C, t_1, t_2) as before, let $\gamma := [g: Y \rightarrow X \times C]$ be a cobordism cycle. Since X is smooth, by the transversality [12, Proposition 3.3.1], we may assume that g is transverse to both i_1 and i_2 . The composition $Y \rightarrow X \times C \rightarrow C$ now has smooth fibres over t_1, t_2 so that we have $-(i_1^* - i_2^*)(\gamma) = \theta(\partial_C(g, t_1, t_2))$. So, if X is smooth, then $\text{Im}(\theta) \supseteq \text{Im}(\theta')$ holds.

Case 2: Suppose X is any scheme. We prove by induction on $\dim X$. Note that every cobordism cycle is a formal sum of cobordism cycles of the form $[f: Y \rightarrow X]$ where Y is smooth and integral, and such f factors uniquely through an irreducible component of X_{red} . Thus, we may reduce to the case when X is integral.

If $\dim X = 0$, then X is smooth so that the statement holds by *Case 1*. Suppose $\dim X > 0$, and assume the statement holds for all lower dimensional schemes in **Sch_k**.

Let $\iota: Z \hookrightarrow X$ be the singular locus and let $U := X \setminus Z$ be the open complement. Using Hironaka's resolution of singularities, we can find a projective morphism $\pi: \tilde{X} \rightarrow X$ that is an isomorphism over U such that the inverse image of Z is a strict normal crossing divisor. Let $[g: Y \rightarrow X \times C] \in \omega_*(X \times C)$, and let $t_1, t_2 \in C(k)$ be two distinct

points. Consider the diagram:

$$\begin{array}{ccccccc}
 E & \hookrightarrow & \tilde{Y} & \xrightarrow{\tilde{g}} & \tilde{X} \times C & \hookrightarrow & U \times C \\
 \downarrow & & \downarrow \mu & \nearrow f' & \downarrow \pi_C & & \parallel \\
 W & \xhookrightarrow{j} & Y & \xrightarrow{g} & X \times C & \hookrightarrow & U \times C \\
 & \searrow & \downarrow j & & \downarrow \iota_C & & \\
 & & W & \xrightarrow{g'} & Z \times C, & &
 \end{array}$$

where $W := g^{-1}(Z \times C)$, g' is the restriction of g on W , f is the rational map $\pi_C^{-1} \circ g$, and μ is a sequence of blow-ups of the indeterminacy of f , which is an isomorphism on the complement of W such that the exceptional divisor E is a strict normal crossing divisor, and such that there is a morphism \tilde{g} making the diagram commute. Moreover, the upper-right and the lower squares are Cartesian.

Let $\alpha := [g: Y \rightarrow X \times C] \in \omega_*(X \times C)$, $\tilde{\alpha} := [\tilde{g}: \tilde{Y} \rightarrow \tilde{X} \times C] \in \omega_*(\tilde{X} \times C)$, and $\beta := [\mu: \tilde{Y} \rightarrow Y] \in \omega_*(Y)$. Recall that for $V \in \mathbf{Sm}_k$, we write $1_V = [\text{Id}: V \rightarrow V] \in \omega_*(V)$. Then as cobordism cycle classes, we have

$$(5.4) \quad \alpha = g_*(1_Y), \quad \tilde{\alpha} = \tilde{g}_*(1_{\tilde{Y}}), \quad \pi_{C*}(\tilde{\alpha}) = g_*\mu_*(1_{\tilde{Y}}) = g_*(\beta).$$

Thus, $\alpha - \pi_{C*}(\tilde{\alpha}) = g_*(1_Y - \beta)$. The blow-up formula [12, Proposition 3.2.4] implies that there is a cobordism cycle $\eta \in \omega_*(W)$ such that $1_Y - \beta = j_*(\eta)$. We thus have

$$(5.5) \quad \alpha - \pi_{C*}(\tilde{\alpha}) = g_*(1_Y - \beta) = g_*j_*(\eta) = \iota_{C*}g'_*(\eta).$$

In particular, we have $i_j^*(\alpha) - i_j^*(\pi_{C*}(\tilde{\alpha})) = i_j^*(\iota_{C*}g'_*(\eta))$ for $j = 1, 2$ so that

$$(5.6) \quad \theta'(\alpha) - \theta'(\pi_{C*}(\tilde{\alpha})) = \theta'(\iota_{C*}g'_*(\eta)).$$

On the other hand, in the Cartesian diagrams below whose rows are regular embeddings,

$$\begin{array}{ccc}
 \tilde{X} \times \{t_j\} & \rightarrow & \tilde{X} \times C \\
 \downarrow \pi & & \downarrow \pi_C \\
 X \times \{t_j\} & \rightarrow & X \times C
 \end{array}
 \quad
 \begin{array}{ccc}
 Z \times \{t_j\} & \rightarrow & Z \times C \\
 \downarrow \iota & & \downarrow \iota_C \\
 X \times \{t_j\} & \rightarrow & X \times C
 \end{array}$$

we can use [12, Proposition 6.5.4] to deduce that $\theta'(\pi_{C*}(\tilde{\alpha})) = \pi_*(\theta'(\tilde{\alpha}))$ and $\theta'(\iota_{C*}g'_*(\eta)) = \iota_*(\theta'(g'_*(\eta)))$. (N.B. : The Tor-independence assumption in [12, Proposition 6.5.4] is only to guarantee that pull-backs of regular embeddings are regular embeddings. In our case, the rows are regular embeddings, thus, the proposition applies here.)

Applying this to (5.6), we conclude that

$$(5.7) \quad \theta'(\alpha) = \pi_*(\theta'(\tilde{\alpha})) + \iota_*(\theta'(g'_*(\eta))).$$

By the *Case 1* applied to \tilde{X} , we have that $\theta'(\tilde{\alpha}) \in \theta(\mathcal{R}_*^{\text{alg}}(\tilde{X}))$ so that $\pi_*(\theta'(\tilde{\alpha})) \in \pi_*(\theta(\mathcal{R}_*^{\text{alg}}(\tilde{X}))) \subset \theta(\mathcal{R}_*^{\text{alg}}(X))$ by Proposition 4.4. Thus, to show $\theta'(\alpha) \in \theta(\mathcal{R}_*^{\text{alg}}(X))$, it is enough to prove that $\theta'(g'_*(\eta)) \in \theta(\mathcal{R}_*^{\text{alg}}(Z))$. But this holds by the induction hypothesis since $\dim Z < \dim X$. Hence, we have shown that $\text{Im}(\theta) \supseteq \text{Im}(\theta')$ for X . This finishes the proof of the theorem. \square

6. EQUIVALENCE OF Ω_*^{alg} AND ω_*^{alg} AND CONSEQUENCES

The purpose of this section is to prove Theorem 6.2, and to establish some fundamental properties of our cobordism theory.

6.1. The comparison theorem. First, we state an analogue of [13, Lemma 3.2] for algebraic equivalence:

Lemma 6.1. *Let $Y \in \mathbf{Sm}_k$ and let E, F be strict normal crossing divisors on Y that are algebraically equivalent. Then $[E \rightarrow Y] = [F \rightarrow Y]$ in $\Omega_*^{\text{alg}}(Y)$.*

Proof. By [12, Proposition 3.1.9], we have $[E \rightarrow Y] = [Y \rightarrow Y, O_Y(E)]$ and $[F \rightarrow Y] = [Y \rightarrow Y, O_Y(F)]$ in $\Omega_*(Y)$. Via the natural map $\Omega_*(Y) \rightarrow \Omega_*^{\text{alg}}(Y)$, these equalities still hold in $\Omega_*^{\text{alg}}(Y)$. It follows from the relation (Equiv) of Definition 3.1 and Lemma 2.4 that $[Y \rightarrow Y, O_Y(E)] = [Y \rightarrow Y, O_Y(F)]$ in $\Omega_*^{\text{alg}}(Y)$. Hence $[E \rightarrow Y] = [F \rightarrow Y]$ in $\Omega_*^{\text{alg}}(Y)$. \square

Theorem 6.2. *For $X \in \mathbf{Sch}_k$, there is a canonical isomorphism $\Omega_*^{\text{alg}}(X) \simeq \omega_*^{\text{alg}}(X)$.*

Proof. We first define a map $\vartheta_X^{\text{alg}}: \omega_*^{\text{alg}}(X) \rightarrow \Omega_*^{\text{alg}}(X)$. We let $\vartheta_X^{\text{alg}}: \mathcal{M}_*(X)^+ \rightarrow \Omega_*^{\text{alg}}(X)$ be given by $\vartheta_X^{\text{alg}}([f: Y \rightarrow X]_{\omega^{\text{alg}}}) = [f: Y \rightarrow X]_{\Omega^{\text{alg}}}$. We need to show that ϑ_X^{alg} kills the algebraic double-point relations.

So let (g, p, ζ) be an algebraic double-point cobordism given by a projective $g: Y \rightarrow X \times C$, where C is a smooth projective curve. It is enough to show that $\partial_C(g, p, \zeta)$ vanishes in $\Omega_*^{\text{alg}}(X)$. Let $f := pr_1 \circ g$ and $\pi := pr_2 \circ g$. Since

$$\partial_C(g, p, \zeta) = f_*([Y_\zeta \rightarrow Y] - [A \rightarrow Y] - [B \rightarrow Y] + [\mathbb{P}(\pi) \rightarrow Y]),$$

it suffices to show that the relation

$$(6.1) \quad [Y_\zeta \rightarrow Y] - [A \rightarrow Y] - [B \rightarrow Y] + [\mathbb{P}(\pi) \rightarrow Y] = 0$$

holds in $\Omega_*^{\text{alg}}(Y)$.

We apply the equation (5.2) to the divisor $E := A + B$ on Y to obtain

$$(6.2) \quad [E \rightarrow Y] - [A \rightarrow Y] - [B \rightarrow Y] + [\mathbb{P}(\pi) \rightarrow Y] = 0 \in \Omega_*^{\text{alg}}(Y).$$

On the other hand, the divisor E is algebraically equivalent to the divisor Y_ζ and hence by Lemma 6.1, we also have the equality $[E \rightarrow Y] = [Y_\zeta \rightarrow Y] \in \Omega_*^{\text{alg}}(Y)$. Combining this with (6.2), we obtain (6.1). Thus, the map $\vartheta_X^{\text{alg}}: \mathcal{M}_*(X)^+ \rightarrow \Omega_*^{\text{alg}}(X)$ descends to give $\vartheta_X^{\text{alg}}: \omega_*^{\text{alg}}(X) \rightarrow \Omega_*^{\text{alg}}(X)$.

To define the inverse $\tau_X^{\text{alg}}: \Omega_*^{\text{alg}}(X) \rightarrow \omega_*^{\text{alg}}(X)$ of ϑ_X^{alg} , we consider the diagram

$$(6.3) \quad \begin{array}{ccccc} \Omega_*(X) & \xrightarrow{\Phi_X} & \Omega_*^{\text{alg}}(X) & \longrightarrow & 0 \\ \tau_X \downarrow & & \downarrow & & \\ \bigoplus_{(C, t_1, t_2)} \omega_*(X \times C) & \xrightarrow{i_1^* - i_2^*} & \omega_*(X) & \xrightarrow{\Psi_X} & \omega_*^{\text{alg}}(X) \longrightarrow 0, \end{array}$$

where the bottom row is exact by Theorem 5.1 and the isomorphism τ_X is from [13, §11.1].

Let θ' be the sum of the maps $i_1^* - i_2^*$. We need to show that $\tau_X(\ker(\Phi_X)) \subseteq \text{Im}(\theta')$ in order to define τ_X^{alg} . By Proposition 3.13, $\ker(\Phi_X)$ is generated by cobordism cycles α of the form $[f: Y \rightarrow X, L_1] - [f: Y \rightarrow X, L_2]$ such that $L_1 \sim L_2$. So, it is enough to show that $\tau_X(\alpha) \in \text{Im}(\theta')$ for such α . We can write $\alpha = f_*(\tilde{c}_1(L_1)(1_Y) - \tilde{c}_1(L_2)(1_Y))$. Applying [12, Theorem 6.5.12] (the Tor-independence assumptions hold by Lemma 6.3

below) to the Cartesian square

$$\begin{array}{ccc} Y \times \{t_j\} & \xrightarrow{i_j} & Y \times C \\ f \downarrow & & f_C \downarrow \\ X \times \{t_j\} & \xrightarrow{i_j} & X \times C, \end{array}$$

we deduce that θ' respects projective push-forwards. Since τ_X also respects projective push-forwards, we may replace X by Y , f by Id_X , and α by $\tilde{c}_1(L_1)(1_Y) - \tilde{c}_1(L_2)(1_Y)$.

Since $L_1 \sim L_2$, there exists a smooth projective curve C , two distinct points $t_1, t_2 \in C(k)$ and a line bundle \mathcal{L} on $X \times C$ such that $\mathcal{L}|_{X \times \{t_j\}} \simeq L_i$ for $j = 1, 2$. We can then write

$$[\text{Id}_X: X \rightarrow X, L_j] = \tilde{c}_1(L_j)(1_X) = (\tilde{c}_1(i_j^*(\mathcal{L})) \circ i_j^*)(1_{X \times C}) = (i_j^* \circ \tilde{c}_1(\mathcal{L}))(1_{X \times C})$$

where the last equality follows from [12, Lemma 7.4.1 (2)].

In particular, we see that $\alpha = i_1^*(\tilde{\alpha}) - i_2^*(\tilde{\alpha})$, where $\tilde{\alpha} = [\text{Id}: X \times C \rightarrow X \times C, \mathcal{L}]$. That is, $\tau_X(\alpha) = \theta'(\tilde{\alpha})$. This shows that $\tau_X(\ker(\Phi_X)) \subseteq \text{Im}(\theta')$ and it defines τ_X^{alg} such that the above diagram commutes.

Both $\omega_*^{\text{alg}}(X)$ and $\Omega_*^{\text{alg}}(X)$ are generated by cobordism cycles of the form $[f: Y \rightarrow X]$, and for those cycles, τ^{alg} and ϑ_X^{alg} are inverses of each other. This proves the theorem. \square

We used the following basic lemma in the proof of the above. We shall use it again to prove Theorem 6.5.

Lemma 6.3. *Let T be a smooth scheme and let $W \subset T$ be a smooth closed subscheme. Then for any morphism $f: V' \rightarrow V$ in \mathbf{Sch}_k , the schemes $V' \times T$ and $V \times W$ are Tor-independent over $V \times T$.*

In particular, if C is a smooth curve and $\{t\}$ is a point in $C(k)$, then for any morphism $f: Y \rightarrow X$ in \mathbf{Sch}_k , the schemes $Y \times C$ and $X \times \{t\}$ are Tor-independent over $X \times C$.

Proof. Since the first assertion is local on V and T , by shrinking them to small enough affine open subschemes if necessary, we may assume that both are affine such that $W \subseteq V$ is a complete intersection subscheme. In particular, there is a finite resolution $\mathcal{F}_\bullet \rightarrow \mathcal{O}_W$ by free \mathcal{O}_T -modules of finite rank. This in turn shows that $\mathcal{F}_\bullet \otimes_k \mathcal{O}_V \rightarrow \mathcal{O}_{V \times W}$ is a finite free resolution of $\mathcal{O}_{V \times W}$ as $\mathcal{O}_{V \times T}$ -module.

Since $(\mathcal{F}_\bullet \otimes_k \mathcal{O}_V) \otimes_{\mathcal{O}_{V \times T}} \mathcal{O}_{V' \times T} \simeq \mathcal{F}_\bullet \otimes_k \mathcal{O}_{V'}$, we have

$$\text{Tor}_i^{\mathcal{O}_{V \times T}}(\mathcal{O}_{V \times W}, \mathcal{O}_{V' \times T}) = \mathcal{H}_i(\mathcal{F}_\bullet \otimes_k \mathcal{O}_{V'}) = 0$$

for $i > 0$. The second assertion is a special case of the first. \square

As an immediate consequence of Theorems 5.1 and 6.2, we obtain:

Theorem 6.4. *For $X \in \mathbf{Sch}_k$, there is an exact sequence:*

$$\bigoplus_{(C, t_1, t_2)} \Omega_*(X \times C) \xrightarrow{i_1^* - i_2^*} \Omega_*(X) \xrightarrow{\Phi_X} \Omega_*^{\text{alg}}(X) \rightarrow 0,$$

where (C, t_1, t_2) runs over the equivalence classes of triples consisting of a smooth projective connected curve C and two distinct points $t_1, t_2 \in C(k)$, and i_j^* is the pull-back via the closed immersion $i_j: X \times \{t_j\} \rightarrow X \times C$ for $j = 1, 2$.

6.2. Fundamental properties of Ω_*^{alg} . We now prove some important properties of our cobordism theory.

Theorem 6.5 (Localization sequence). *Given a closed immersion $Y \hookrightarrow X$ in \mathbf{Sch}_k with complement U , there is an exact sequence*

$$\Omega_*^{\text{alg}}(Y) \rightarrow \Omega_*^{\text{alg}}(X) \rightarrow \Omega_*^{\text{alg}}(U) \rightarrow 0.$$

Proof. Let $\iota: Y \rightarrow X$ and $j: U \rightarrow X$ be the inclusions. Consider the diagram

$$(6.4) \quad \begin{array}{ccccc} \bigoplus_{(C,t_1,t_2)} \Omega_*(Y \times C) & \xrightarrow{i_1^* - i_2^*} & \Omega_*(Y) & \xrightarrow{\Phi_Y} & \Omega_*^{\text{alg}}(Y) \longrightarrow 0 \\ \downarrow \iota_{C*} & & \downarrow \iota_* & & \downarrow \iota_* \\ \bigoplus_{(C,t_1,t_2)} \Omega_*(X \times C) & \xrightarrow{i_1^* - i_2^*} & \Omega_*(X) & \xrightarrow{\Phi_X} & \Omega_*^{\text{alg}}(X) \longrightarrow 0 \\ \downarrow j_C^* & & \downarrow j^* & & \downarrow j^* \\ \bigoplus_{(C,t_1,t_2)} \Omega_*(U \times C) & \xrightarrow{i_1^* - i_2^*} & \Omega_*(U) & \xrightarrow{\Phi_U} & \Omega_*^{\text{alg}}(U) \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0, \end{array}$$

where $\iota_C: Y \times C \rightarrow X \times C$ and $j_C: U \times C \rightarrow X \times C$ are the induced inclusions. Here, the rows are exact by Theorem 6.4 and the first two columns are exact by [12, Theorem 3.2.7]. This diagram commutes: the bottom left square commutes by the composition law of the pull-backs. The top and the bottom right squares commute by the naturality of the maps $\Phi_{(-)}$. For the top left square, consider the Cartesian square:

$$\begin{array}{ccc} Y \times \{t_j\} & \xrightarrow{1_Y \times i_j} & Y \times C \\ \iota \downarrow & & \downarrow \iota_C \\ X \times \{t_j\} & \xrightarrow{1_X \times i_j} & X \times C, \end{array}$$

where the horizontal maps are l.c.i. morphisms and the vertical maps are closed immersions. By Lemma 6.3, $1_X \times i_j$ and ι_C are Tor-independent. Hence by [12, Theorem 6.5.12], we have $(1_X \times i_j)^* \circ \iota_{C*} = \iota_* \circ (1_Y \times i_j)^*$, which implies that $((1_X \times i_1)^* - (1_X \times i_2)^*) \circ \iota_{C*} = \iota_* \circ ((1_Y \times i_1)^* - (1_Y \times i_2)^*)$. This means that the top left square of (6.4) commutes. Thus, we have shown that the diagram (6.4) commutes. A simple diagram chase now shows that the third column is also exact. This proves the theorem. \square

Theorem 6.6 (\mathbb{A}^1 -homotopy Invariance). *Let $X \in \mathbf{Sch}_k$ and let $p: V \rightarrow X$ be a torsor under a vector bundle over X of rank n . Then the map $p^*: \Omega_*^{\text{alg}}(X) \rightarrow \Omega_{*+n}^{\text{alg}}(V)$ is an isomorphism.*

Proof. We consider the commutative diagram

$$\begin{array}{ccccc} \bigoplus_{(C,t_1,t_2)} \Omega_*(X \times C) & \xrightarrow{i_1^* - i_2^*} & \Omega_*(X) & \xrightarrow{\Phi_X} & \Omega_*^{\text{alg}}(X) \longrightarrow 0 \\ \downarrow p^* & & \downarrow p^* & & \downarrow p^* \\ \bigoplus_{(C,t_1,t_2)} \Omega_{*+n}(V \times C) & \xrightarrow{i_1^* - i_2^*} & \Omega_{*+n}(V) & \xrightarrow{\Phi_V} & \Omega_{*+n}^{\text{alg}}(V) \longrightarrow 0, \end{array}$$

where the rows are exact by Theorem 6.4. The first two vertical arrows are isomorphisms by [12, Theorem 3.6.3]. Hence, so is the third arrow by diagram chasing. \square

Using the projective bundle formula [12, Theorem 3.5.4] for Ω_* , the argument of the proof of Theorem 6.6 can be repeated in verbatim with V replaced by $\mathbb{P}(V)$ to prove the following projective bundle formula for our cobordism theory.

Theorem 6.7 (Projective bundle formula). *Let $X \in \mathbf{Sch}_k$ and let E be a rank $n + 1$ vector bundle on X . Then, we have $\bigoplus_{j=0}^n \Omega_{*-n+j}^{\text{alg}}(X) \xrightarrow{\cong} \Omega_*^{\text{alg}}(\mathbb{P}(E))$.*

7. Ω_{alg}^* AS AN ORIENTED COHOMOLOGY THEORY

Recall from [12, Definition 1.1.2] that an oriented cohomology theory A^* on \mathbf{Sm}_k is an additive contravariant functor to the category of commutative graded rings with unit, such that A^* has push-forward maps for projective morphisms and it satisfies the \mathbb{A}^1 -homotopy invariance and projective bundle formula. Moreover, the push-forward and the pull-back maps commute in a Cartesian diagram of transverse morphisms.

On the bigger category \mathbf{Sch}_k , there is a notion of an oriented Borel-Moore homology theory (see [12, Definition 5.1.3]). This requires some similar axioms, but a nontrivial one is the existence of pull-backs for l.c.i. morphisms. This ensures that an oriented Borel-Moore homology theory on \mathbf{Sch}_k restricted onto \mathbf{Sm}_k gives an oriented cohomology theory.

Our goal in this section is to conclude that Ω_{alg}^* is an oriented cohomology theory on \mathbf{Sm}_k and Ω_*^{alg} is an oriented Borel-Moore homology theory on \mathbf{Sch}_k .

7.1. Pull-back via l.c.i. morphisms. By Definition 2.2, one can pull-back cobordism cycles via *smooth* quasi-projective morphisms. One further step is needed to turn Ω_*^{alg} into an oriented Borel-Moore homology : to show that one can pull-back also via l.c.i. morphisms $f: X \rightarrow Y$ for $X, Y \in \mathbf{Sch}_k$. Recall that $f: X \rightarrow Y$ is an l.c.i. morphism if it factors as the composition $f = q \circ i: X \rightarrow P \rightarrow Y$, where i is a regular embedding and q is a smooth quasi-projective morphism. Since we have q^* already, defining i^* is the first technical issue to resolve. We shall demonstrate the existence of such pull-backs on Ω_*^{alg} using Proposition 3.13 and the analogous construction for the algebraic cobordism in [12, §5, 6].

Recall from [6, Definition 2.2.1] that a *pseudo-divisor* D on a scheme X is a triple $D = (Z, \mathcal{L}, s)$, where $Z \subset X$ is a closed subset, \mathcal{L} is an invertible sheaf on X , and s is a section of \mathcal{L} on X such that the support of the zero scheme of s is contained in Z . We call Z , *the support of D* and write it as $|D|$. We call the zero scheme $\{s = 0\}$, *the divisor of D* and write it as $\text{Div}(D)$.

Given $X \in \mathbf{Sch}_k$ and a pseudo-divisor D on X , Levine and Morel defined in [12, §6.1.2] a graded group $\Omega_*(X)_D$ with a natural map $\theta_X: \Omega_*(X)_D \rightarrow \Omega_*(X)$, which is an isomorphism by [12, Theorem 6.4.12]. Roughly speaking, this is the group on which the “intersection product” by the pseudo-divisor D is well-defined so that we have a map $D(-): \Omega_*(X)_D \rightarrow \Omega_{*-1}(|D|)$. (See § 11 for the definitions of $\Omega_*(X)_D$ and $D(-)$.) This yields

$$(7.1) \quad i_D^*: \Omega_*(X) \xrightarrow[\cong]{\theta_X^{-1}} \Omega_*(X)_D \xrightarrow{D(-)} \Omega_{*-1}(|D|).$$

It follows from Proposition 3.13 and Lemma 11.6 that i_D^* descends to

$$(7.2) \quad i_D^*: \Omega_*^{\text{alg}}(X) \rightarrow \Omega_{*-1}^{\text{alg}}(|D|).$$

7.1.1. *Gysin map for regular embedding.* Let $\iota_Z: Z \rightarrow X$ be a regular embedding of codimension d in \mathbf{Sch}_k . We use (7.2) and the technique of the deformation to the normal bundle to define the pull-back map $\iota_Z^*: \Omega_{**}^{\text{alg}}(X) \rightarrow \Omega_{*-d}^{\text{alg}}(Z)$, that we call *the Gysin map* for the cobordism classes. Without going into the full construction of the deformation to the normal bundle, we recall here only the necessary summary from [12, §6.5.2 (6.10)]:

Proposition 7.1. *Let $\iota_Z: Z \rightarrow X$ be a regular embedding in \mathbf{Sch}_k . Then, there exists a scheme $U \in \mathbf{Sch}_k$, a closed immersion $i_N: N \rightarrow U$ of codimension one, a surjective morphism $\mu: U \rightarrow X \times \mathbb{P}^1$, and its restriction $\mu_N: N \rightarrow Z \times 0$, that form the commutative diagram*

$$\begin{array}{ccc} N & \xrightarrow{i_N} & U \\ \mu_N \downarrow & & \downarrow \mu \\ Z \times 0 & \xrightarrow{\text{Id} \times 0} Z \times \mathbb{P}^1 & \xrightarrow{\iota_Z \times \text{Id}} X \times \mathbb{P}^1 \end{array}$$

such that

- (1) N is isomorphic to the normal vector bundle $N_{Z/X}$ of Z in X over Z under the identification $Z = Z \times 0$, and
- (2) the restriction $\mu: U \setminus N \rightarrow X \times (\mathbb{P}^1 \setminus \{0\})$ is an isomorphism of schemes.

We have the following analogue of [12, Lemma 6.5.2]:

Lemma 7.2. *The composition $i_N^* \circ i_{N*}: \Omega_{*+1}^{\text{alg}}(N) \rightarrow \Omega_{*+1}^{\text{alg}}(U) \rightarrow \Omega_*^{\text{alg}}(N)$ is zero, where i_{N*} is the push-forward via the closed immersion i_N , and i_N^* is the pull-back by the divisor N defined in (7.2).*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \Omega_{*+1}(N) & \xrightarrow{i_N^* \circ i_{N*}} & \Omega_*(N) \\ \downarrow & & \downarrow \\ \Omega_{*+1}^{\text{alg}}(N) & \xrightarrow{i_N^* \circ i_{N*}} & \Omega_*^{\text{alg}}(N), \end{array}$$

where the vertical maps are the natural surjections. Since the top map on the algebraic cobordism is zero by [12, Lemma 6.5.2], the bottom map is also zero. \square

By Theorem 6.5, we have the localization exact sequence

$$\Omega_{*+1}^{\text{alg}}(N) \xrightarrow{i_{N*}} \Omega_{*+1}^{\text{alg}}(U) \xrightarrow{j^*} \Omega_{*+1}^{\text{alg}}(U \setminus N) \rightarrow 0,$$

that gives an isomorphism

$$(7.3) \quad (j^*)^{-1}: \Omega_{*+1}^{\text{alg}}(U \setminus N) \rightarrow \frac{\Omega_{*+1}^{\text{alg}}(U)}{i_{N*}(\Omega_{*+1}^{\text{alg}}(N))}.$$

Combining (7.3) with Lemma 7.2, we see that the composition

$$(7.4) \quad \alpha: \Omega_{*+1}^{\text{alg}}(U \setminus N) \xrightarrow{(j^*)^{-1}} \frac{\Omega_{*+1}^{\text{alg}}(U)}{i_{N*}(\Omega_{*+1}^{\text{alg}}(N))} \xrightarrow{i_N^*} \Omega_*^{\text{alg}}(N)$$

is well-defined.

Definition 7.3. For a regular embedding $\iota_Z: Z \rightarrow X$ of codimension d in \mathbf{Sch}_k , the Gysin morphism $\iota_Z^*: \Omega_*^{\text{alg}}(X) \rightarrow \Omega_{*-d}^{\text{alg}}(Z)$ is defined to be the composition

$$\Omega_*^{\text{alg}}(X) \xrightarrow{pr_1^*} \Omega_{*+1}^{\text{alg}}(X \times (\mathbb{P}^1 \setminus \{0\})) \xrightarrow[\simeq]{\mu^*} \Omega_{*+1}^{\text{alg}}(U \setminus N) \xrightarrow{\alpha} \Omega_*^{\text{alg}}(N) \xrightarrow[\simeq]{(\mu_N^*)^{-1}} \Omega_{*-d}^{\text{alg}}(Z),$$

where pr_1 is the projection, μ is the isomorphism of Proposition 7.1(2), α is the map in (7.4), and $\mu_N: N \rightarrow Z$ is the normal bundle of Proposition 7.1(1) so that μ_N^* is an isomorphism by Theorem 6.6.

We have the following basic properties for the Gysin maps on Ω_*^{alg} that can be easily deduced from [12, Lemmas 6.5.6, 6.5.7, Theorem 6.5.8] combined with the surjectivity of $\Phi_X: \Omega_*(X) \rightarrow \Omega_*^{\text{alg}}(X)$, as in the proof of Lemma 7.2. We skip the details:

Proposition 7.4. *The Gysin maps on Ω_*^{alg} satisfy the following:*

(1) *Let $\iota: Z \rightarrow X$ be a regular embedding of codimension one. Then, as operators $\Omega_*^{\text{alg}}(X) \rightarrow \Omega_{*-1}^{\text{alg}}(Z)$, the pull-back $Z(-)$ by the divisor Z is identical to the Gysin pull-back ι^* .*

(2) *Let $\iota: Z \rightarrow X$ be a regular embedding, let $p: Y \rightarrow X$ be a smooth quasi-projective morphism, and let $s: Z \rightarrow Y$ be a section of Y over Z . Then, $s^* \circ p^* = \iota^*$.*

(3) *Let $\iota: Z \rightarrow Z'$ and $\iota': Z' \rightarrow X$ be regular embeddings. Then, $(\iota' \circ \iota)^* = \iota^* \circ \iota'^*$.*

7.1.2. Pull-back for l.c.i. morphisms. Let $f: X \rightarrow Y$ be an l.c.i. morphism in \mathbf{Sch}_k with a factorization $f = p \circ i: X \rightarrow P \rightarrow Y$, where p is smooth quasi-projective and i is a regular embedding. We have p^* by Definition 2.2, and we have the Gysin pull-back i^* by Definition 7.3. So, one wishes to define f^* by taking the composition $i^* \circ p^*$. To show that this definition is meaningful, one needs to know that if $p_1 \circ i_1 = p_2 \circ i_2$ are two such factorizations, then $i_1^* \circ p_1^* = i_2^* \circ p_2^*$. However, this fact follows at once from such an equality on the level of algebraic cobordism, as shown in [12, Lemma 6.5.9], and from the surjection $\Phi_-: \Omega_*(-) \rightarrow \Omega_*^{\text{alg}}(-)$. Thus we have:

Definition 7.5. Let $f: X \rightarrow Y$ be an l.c.i. morphism that has a factorization $f = p \circ i: X \rightarrow P \rightarrow Y$, where i is a regular embedding and p is smooth quasi-projective. The pull-back f^* on $\Omega_*^{\text{alg}}(Y)$ is defined to be $i^* \circ p^*$.

One has the following properties of the l.c.i. pull-backs on Ω_*^{alg} as proven for Ω_* in [12, Theorems 6.5.11, 6.5.12, 6.5.13]. The proof follows immediately from *ibids.* and we omit the arguments.

Theorem 7.6. *The pull-backs on Ω_*^{alg} via l.c.i. morphisms have the following properties:*

(1) *If $f_1: X \rightarrow Y$, $f_2: Y \rightarrow Z$ are l.c.i. morphisms in \mathbf{Sch}_k , then $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$.*

(2) *Suppose $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ are Tor-independent morphisms in \mathbf{Sch}_k , where f is l.c.i. and g is projective. Then, for the Cartesian square*

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{pr_2} & Y \\ pr_1 \downarrow & & g \downarrow \\ X & \xrightarrow{f} & Z, \end{array}$$

we have $f^ \circ g_* = pr_{1*} \circ pr_2^*$.*

(3) *Let $f_i: X_i \rightarrow Y_i$ for $i = 1, 2$ be two l.c.i. morphisms in \mathbf{Sch}_k . Then, for $\eta_i \in \Omega_*^{\text{alg}}(Y_i)$ with $i = 1, 2$, we have $(f_1 \times f_2)^*(\eta_1 \times \eta_2) = f_1^*(\eta_1) \times f_2^*(\eta_2)$.*

Corollary 7.7. *Let $f: X \rightarrow Y$ be any morphism of smooth schemes. Then, there is a well-defined pull-back $f^*: \Omega_{\text{alg}}^*(Y) \rightarrow \Omega_{\text{alg}}^*(X)$. If $f: X \rightarrow Y$, $g: Y \rightarrow Z$ are any morphisms of smooth schemes, then $(g \circ f)^* = f^* \circ g^*$.*

Proof. Any morphism $f: X \rightarrow Y$ of smooth schemes is an l.c.i. morphism, with a factorization $f = pr_2 \circ \Gamma_f: X \rightarrow X \times Y \rightarrow Y$. The rest follows immediately. \square

The main results proven in § 6.2 and § 7.1 can now be summarized as follows:

Theorem 7.8. *The theory Ω_{alg}^* is an oriented cohomology theory on \mathbf{Sm}_k that respects algebraic equivalence, and it is universal among such theories. In other words, for any oriented cohomology theory A^* that respects algebraic equivalence, there exists a unique morphism of oriented cohomology theories $\theta: \Omega_{\text{alg}}^* \rightarrow A^*$ on \mathbf{Sm}_k .*

Similarly, the theory Ω_^{alg} is an oriented Borel-Moore homology theory on \mathbf{Sch}_k that respects algebraic equivalence, and it is universal among such theories.*

8. CONNECTIONS TO ALGEBRAIC COBORDISM, CHOW GROUPS AND K -THEORY

In this section, we study how our cobordism theory $\Omega_*^{\text{alg}}(X)$ is related with the Chow groups $\text{CH}_*^{\text{alg}}(X)$ modulo algebraic equivalence and the semi-topological K -groups $K_0^{\text{semi}}(X)$ and $G_0^{\text{semi}}(X)$. We shall also show that our cobordism theory agrees with the algebraic cobordism theory with finite coefficients.

8.1. Connection with Chow groups and K -theory.

Theorem 8.1. *For $X \in \mathbf{Sch}_k$, there is a natural map $\Omega_*^{\text{alg}}(X) \rightarrow \text{CH}_*^{\text{alg}}(X)$ that induces an isomorphism $\Omega_*^{\text{alg}}(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \xrightarrow{\cong} \text{CH}_*^{\text{alg}}(X)$.*

Proof. We consider the commutative diagram

$$(8.1) \quad \begin{array}{ccccccc} \bigoplus_{(C, t_1, t_2)} \Omega_*(X \times C) & \xrightarrow{i_1^* - i_2^*} & \Omega_*(X) & \longrightarrow & \Omega_*^{\text{alg}}(X) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus_{(C, t_1, t_2)} \text{CH}_*(X \times C) & \xrightarrow{i_1^* - i_2^*} & \text{CH}_*(X) & \longrightarrow & \text{CH}_*^{\text{alg}}(X) & \longrightarrow & 0, \end{array}$$

where the top row is exact by Theorem 6.4. It follows from the definition of algebraic equivalence of algebraic cycles in [6, Definition 10.3] and the proof of Lemma 2.4 that the bottom row is also exact (see [6, Example 10.3.2] when k is algebraically closed). The existence of the first two vertical maps and their commutativity follow from the universal property of Ω_* . This immediately yields a natural map $\Omega_*^{\text{alg}}(X) \rightarrow \text{CH}_*^{\text{alg}}(X)$.

Moreover, the top row remains exact after applying the functor $-\otimes_{\mathbb{L}_*} \mathbb{Z}$ and the first two vertical maps after tensoring are isomorphisms by [12, Theorem 4.5.1]. Thus, the last vertical map after tensoring is also an isomorphism. \square

Remark 8.2. By Theorems 7.8, 8.1, and [12, Theorem 1.2.2], we see that CH_{alg}^* is universal among oriented cohomology theories on \mathbf{Sm}_k whose Chern class operations are additive, i.e., $\tilde{c}_1(L_1 \otimes L_2) = \tilde{c}_1(L_1) + \tilde{c}_1(L_2)$ and respect algebraic equivalence.

For $X \in \mathbf{Sch}_k$, let $K_0(X)$ (resp. $G_0(X)$) be the Grothendieck group of coherent locally free sheaves (resp. coherent sheaves) on X . Recall from [5, Definition 1.1] that the semi-topological K -group $K_0^{\text{semi}}(X)$ (resp. $G_0^{\text{semi}}(X)$) is the quotient by the subgroup generated by the images of the l.c.i. pull-backs $i_1^* - i_2^*: K_0(X \times C) \rightarrow K_0(X)$ (resp. $i_1^* - i_2^*: G_0(X \times C) \rightarrow G_0(X)$) over the equivalence classes of the triples (C, t_1, t_2) . When X is smooth, we have $K_0^{\text{semi}}(X) \xrightarrow{\cong} G_0^{\text{semi}}(X)$. We have the following analogue of [12, Corollary 4.2.12].

Theorem 8.3. *Let $X \in \mathbf{Sch}_k$ and let β be a formal symbol of degree -1 . Then, there is a natural map $\Omega_*^{\text{alg}}(X) \rightarrow G_0^{\text{semi}}(X)[\beta, \beta^{-1}]$ which induces an isomorphism $\Omega_*^{\text{alg}}(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}] \xrightarrow{\sim} G_0^{\text{semi}}(X)[\beta, \beta^{-1}]$.*

Proof. This follows from the definition of $G_0^{\text{semi}}(X)$ above, Theorem 6.4, together with [12, Corollary 4.2.12] (if X is smooth) and [2, Theorem 1.5] (if X is not smooth) by repeating the arguments in the proof of Theorem 8.1 in verbatim. The only change is that we have to apply the functor $- \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}]$ instead of $- \otimes_{\mathbb{L}_*} \mathbb{Z}$ to the exact sequence similar to that of (8.1), where the bottom row consists of G_0 instead of CH . \square

8.2. Comparison with algebraic cobordism with finite coefficients. By [4, Corollary 3.8], we know that with finite coefficients, the algebraic and the semi-topological K -theories of complex projective varieties coincide. The following is the cobordism analogue of this agreement.

Theorem 8.4. *Let $X \in \mathbf{Sch}_k$ and let $m \geq 1$ be an integer. Then the natural map $\Phi_X \otimes \mathbb{Z}/m: \Omega_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m \rightarrow \Omega_*^{\text{alg}}(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m$ is an isomorphism.*

Proof. Using [13, Theorem 1] and Theorem 6.2, we can identify $\Omega_*(X)$ and $\Omega_*^{\text{alg}}(X)$ with $\omega_*(X)$ and $\omega_*^{\text{alg}}(X)$, respectively. In the diagram (5.3), it suffices to show that $\text{Im}(\theta)$ in $\omega_*(X)$ is divisible.

Let (g, p, ζ) be a double-point cobordism with a projective $g: Y \rightarrow X \times C$, two points $p, \zeta \in C(k)$ and $\pi = pr_2 \circ g$ such that $\pi^{-1}(p) = A \cup B$ (see Definition 4.2).

Let $\alpha: = [Y_{\zeta} \rightarrow Y] - [A \rightarrow Y] - [B \rightarrow Y] + [\mathbb{P}(\pi) \rightarrow Y]$ in $\omega_*(Y)$. Set $f: = pr_1 \circ g: Y \rightarrow X$. Since $\partial_C(g, p, \zeta) = f_*(\alpha)$, it suffices to show that α is divisible in $\omega_*(Y)$. An application of (5.2) to the divisor $E: = A + B$ shows that $[A \rightarrow Y] + [B \rightarrow Y] - [\mathbb{P}(\pi) \rightarrow Y] = [E \rightarrow Y] = \pi^*([\{p\} \rightarrow C])$. We also have $[Y_{\zeta} \rightarrow Y] = \pi^*([\{\zeta\} \rightarrow C])$. Thus, $\alpha = \pi^*([\{\zeta\} \rightarrow C] - [\{p\} \rightarrow C])$ and it reduces to proving that the class $\beta: = [\{\zeta\} \rightarrow C] - [\{p\} \rightarrow C]$ is divisible in $\omega_0(C)$.

By [12, Lemma 4.5.3], the natural map $\omega_0(C) \rightarrow \text{CH}_0(C)$ is an isomorphism and the image of β in $\text{CH}_0(C)$ is $[\{\zeta\}] - [\{p\}]$, which lies in $\text{Pic}^0(C)$. Since $\text{Pic}^0(C)$ is an abelian variety, the group $\text{Pic}^0(C)(k)$ is divisible. This completes the proof. \square

9. COMPUTATIONS OF Ω_*^{alg} AND QUESTIONS ON FINITE GENERATION

It is usually not easy to compute Ω_* . For the point $X = \text{Spec}(k)$, Levine and Morel [12, Theorem 1.2.7] showed that the natural map $\mathbb{L}_* \rightarrow \Omega_*(k)$ is an isomorphism. In this section, we focus on some computational aspects of Ω_*^{alg} .

9.1. Comparison with the complex cobordism. We refer to [16] or [18] for the definition and basic properties of the complex cobordism theory MU^* for locally compact Hausdorff topological spaces. We only mention here that $\text{MU}^*(X)$ is generated by $[f: Y \rightarrow X]$, where f is proper and Y is a weakly complex real manifold under certain “bordism relations”.

Proposition 9.1. *Given an embedding $\sigma: k \hookrightarrow \mathbb{C}$, there is a natural transformation $\theta^{\text{alg}}: \Omega_{\text{alg}}^* \rightarrow \text{MU}^{2*}$ of oriented cohomology theories on \mathbf{Sm}_k that factors the cycle class map $\theta: \Omega^* \rightarrow \text{MU}^{2*}$. This θ^{alg} is a lifting of the cycle class map of Totaro $\text{CH}_{\text{alg}}^*(X) \rightarrow \text{MU}^{2*}(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$ (see [18, Theorem 3.1]).*

Proof. From [12, Example 1.2.10], we have a morphism $\theta: \Omega^* \rightarrow \text{MU}^{2*}$ of oriented cohomology theories on \mathbf{Sm}_k . Hence by Theorem 7.8, it suffices to show that for any

$X \in \mathbf{Sm}_k$ and algebraically equivalent line bundles L_1 and L_2 on X , one has $\tilde{c}_1(L_1) = \tilde{c}_1(L_2): \mathrm{MU}^*(X_\sigma) \rightarrow \mathrm{MU}^{*+2}(X_\sigma)$. We can assume $k = \mathbb{C}$.

Let \mathcal{L} be a line bundle on $X \times C$ for some compact Riemann surface C such that for some points $t_1, t_2 \in C$, we have $L_j = \mathcal{L}|_{X \times \{t_j\}}$ for $j = 1, 2$. Let $i_j: X \times \{t_j\} \rightarrow X \times C$ be the inclusions. Take any differentiable path I in C , diffeomorphic to the unit interval $[0, 1]$, whose end points are t_1 and t_2 . Let $\alpha: X \times I \rightarrow X \times C$ and $\iota_j: X \times \{t_j\} \rightarrow X \times I$ be the inclusions. Note that $\alpha \circ \iota_j = i_j$ for $j = 1, 2$.

Since X is smooth, we have $\tilde{c}_1(L_j)([Y \rightarrow X]) = (\tilde{c}_1(L_j)(1_X)) \cdot [Y \rightarrow X] = c_1(L_j) \cdot [Y \rightarrow X]$, where the first equality comes from [12, (5.2)-5]. On the other hand, we have $c_1(L_j) = i_j^*(c_1(\mathcal{L})) = \iota_j^* \alpha^*(c_1(\mathcal{L}))$. The desired assertion now follows from the fact that $\iota_j^*: \mathrm{MU}^*(X \times I) \rightarrow \mathrm{MU}^*(X)$ is an isomorphism for $j = 1, 2$ because I is contractible. The second assertion follows from the first assertion and Theorem 8.1. \square

9.2. Points.

Proposition 9.2. *The map $\mathbb{L}^* \rightarrow \Omega_{\mathrm{alg}}^*(k)$ is an isomorphism.*

Proof. Composing the isomorphism $\mathbb{L}^* \xrightarrow{\sim} \Omega^*(k)$ with the surjection $\Omega^*(k) \rightarrow \Omega_{\mathrm{alg}}^*(k)$, we see that the map $\mathbb{L}^* \rightarrow \Omega_{\mathrm{alg}}^*(k)$ is surjective. We prove injectivity.

We first prove the injectivity of the map $\mathbb{L}^* \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \Omega_{\mathrm{alg}}^*(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the rational coefficients. Applying Proposition 3.13, we see that $\ker(\Omega^*(k) \rightarrow \Omega_{\mathrm{alg}}^*(k))$ is generated by the cobordism cycles of the form $\alpha = [Y \rightarrow \mathrm{Spec}(k), L] - [Y \rightarrow \mathrm{Spec}(k), M]$, where $L \sim M$ on Y . Since we are working with the rational coefficients, we can use [13, Theorem 1, Corollary 3] to assume that Y is a product of projective spaces. But on such spaces, two line bundles are algebraically equivalent if and only if they are isomorphic. In particular, α is zero already in $\Omega^*(k) \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus, the map $\mathbb{L}^* \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \Omega_{\mathrm{alg}}^*(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective. The injectivity of $\mathbb{L}^* \rightarrow \Omega_{\mathrm{alg}}^*(k)$ now follows because \mathbb{L}^* has no torsion. \square

Recall from [12, Definition 4.4.1] that an oriented Borel-Moore homology theory A_* on \mathbf{Sch}_k is said to be *generically constant*, if for each finitely generated field extension $k \subset F$, the canonical morphism $A_*(k) \rightarrow A_*(F/k)$ is an isomorphism. Here $A_*(F/k)$ is the colimit of $A_{*+\mathrm{tr}_{F/k}}(X)$ over models X for F over k and $\mathrm{tr}_{F/k}$ is the transcendence degree of F over k . Recall that a model for F over k is an integral scheme $X \in \mathbf{Sch}_k$ whose function field is isomorphic to F .

Proposition 9.3. *The cobordism theory Ω_*^{alg} is generically constant.*

Proof. Let \mathcal{C} denote the category of models for F over k . Then, we have a commutative diagram

$$\begin{array}{ccccc} \Omega_*(k) & \xrightarrow{\cong} & \Omega_*^{\mathrm{alg}}(k) & & \\ \eta_F \downarrow \cong & & \eta_F^{\mathrm{alg}} \downarrow & \searrow \cong & \\ \mathrm{colim}_{X \in \mathcal{C}} \Omega_{*+\mathrm{tr}_{F/k}}(X) & \longrightarrow & \mathrm{colim}_{X \in \mathcal{C}} \Omega_{*+\mathrm{tr}_{F/k}}^{\mathrm{alg}}(X) & \longrightarrow & \Omega_*^{\mathrm{alg}}(F). \end{array}$$

We need to show that η_F^{alg} is an isomorphism. It follows from [12, Corollary 4.4.3] that η_F is an isomorphism. Applying Proposition 3.13 to the first horizontal arrow on the bottom, we see that η_F^{alg} is surjective. On the other hand, it follows from Proposition 9.2 that the slanted downward arrow is an isomorphism. This in turn implies that η_F^{alg} must also be injective, and hence an isomorphism. \square

Recall from [6, Example 1.9.1] that a scheme $X \in \mathbf{Sch}_k$ is called *cellular* if it has a filtration $\emptyset = X_{n+1} \subsetneq X_n \subsetneq \cdots \subsetneq X_1 \subsetneq X_0 = X$ by closed subschemes such that each $X_i \setminus X_{i+1}$ is a disjoint union of affine spaces, called *cells*. Basic examples include projective spaces, smooth projective toric varieties, and schemes of type G/P , where P is a parabolic subgroup of a split reductive group G . As a consequence of Proposition 9.2 and a general result of A. Nenashev and K. Zainoulline [14, Theorem 5.9], we can easily compute our cobordism theory for cellular schemes:

Proposition 9.4. *For a cellular scheme $X \in \mathbf{Sch}_k$, the natural map $\Phi_X: \Omega_*(X) \rightarrow \Omega_*^{\text{alg}}(X)$ is an isomorphism. Each of these groups is a free \mathbb{L}_* -module of rank equal to the number of cells.*

Remark 9.5. One may also directly prove Proposition 9.4 by an induction argument on the length of a filtration of X using Theorem 6.5 and [11, Proposition 4.3]. If there is an embedding $\sigma: k \hookrightarrow \mathbb{C}$, Proposition 9.4 can be also deduced from Proposition 3.13, Proposition 9.1 and [8, Theorem 6.1].

9.3. Curves. We next compute the cobordism theory $\Omega_{\text{alg}}^*(X)$ of a smooth curve X . We show that this is a finitely generated \mathbb{L}^* -module. This is usually false for the algebraic cobordism $\Omega^*(X)$ unless X is rational. If $k = \mathbb{C}$, we show that $\Omega_{\text{alg}}^*(X)$ is closely related to the complex cobordism $\text{MU}^*(X(\mathbb{C}))$.

Theorem 9.6. *Let X be a connected smooth curve over a field k . Then,*

- (1) *The \mathbb{L}^* -module $\Omega_{\text{alg}}^*(X)$ is generated by at most 2 elements.*
- (2) *When X is affine, the map $\mathbb{L}^* \rightarrow \Omega_{\text{alg}}^*(X)$ is an isomorphism.*
- (3) *When $k = \mathbb{C}$, the map $\Omega_{\text{alg}}^*(X) \rightarrow \text{MU}^{2*}(X(\mathbb{C}))$ is an isomorphism.*

Proof. We have shown in Theorem 6.5 and Proposition 9.3 that Ω_{alg}^* has the localization property and is generically constant. Hence, it satisfies the generalized degree formula [12, Theorem 4.4.7]. By this degree formula, the cobordism $\Omega_{\text{alg}}^*(X)$ is generated as an \mathbb{L}^* -module by the cobordism cycles $1_X = [X \rightarrow X]$ and $[\{p\} \rightarrow X] = [X \rightarrow X, O_X(p)]$, where p is a closed point of X . Part (1) now follows from the fact that the map $\deg: \text{Pic}(X)/\sim \rightarrow \mathbb{Z}$ is injective.

If X is affine, we choose a smooth compactification $j: X \hookrightarrow \overline{X}$ and set $Z := \overline{X} \setminus X$. This yields an exact sequence

$$\text{CH}^0(Z) \rightarrow \text{Pic}(\overline{X})/\sim \xrightarrow{j^*} \text{Pic}(X)/\sim \rightarrow 0$$

by [6, Example 10.3.4], in which the first map is surjective. In particular, the last term is zero. Thus $\Omega_{\text{alg}}^*(X)$ is generated by 1_X as an \mathbb{L}^* -module, *i.e.*, $\mathbb{L}^* \rightarrow \Omega_{\text{alg}}^*(X)$ is surjective. On the other hand, for a closed point $p \in X$, the composition with the pull-back $\mathbb{L}^* \rightarrow \Omega_{\text{alg}}^*(X) \rightarrow \Omega_{\text{alg}}^*(k(p))$ is an isomorphism by Proposition 9.2. We conclude that the map $\mathbb{L}^* \rightarrow \Omega_{\text{alg}}^*(X)$ is injective and hence an isomorphism. This proves (2).

For (3), we first observe that as $X(\mathbb{C})$ is a topological surface, we have an induced isomorphism

$$(9.1) \quad \text{MU}^*(X(\mathbb{C})) \xrightarrow{\cong} H^*(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{L}^*$$

by [18, Theorem 2.2]. Since the cycle class map of Proposition 9.1 maps $\Omega_{\text{alg}}^*(X)$ into $\text{MU}^{2*}(X(\mathbb{C}))$, we look at only the even degrees. When X is affine, we have $H^2(X(\mathbb{C}), \mathbb{Z}) = 0$ so that $\text{MU}^{2*}(X(\mathbb{C})) = H^0(X(\mathbb{C}), \mathbb{L}^*) = \mathbb{L}^*$. By part (2), the natural map $\Omega_{\text{alg}}^*(X) \rightarrow \text{MU}^{2*}(X(\mathbb{C}))$ is simply the identity map of \mathbb{L}^* .

When X is not affine (thus, projective), we get $H^i(X(\mathbb{C}), \mathbb{L}^*) = \mathbb{L}^*$ for both $i = 0$ and 2 , and this yields $\mathrm{MU}^{2*}(X(\mathbb{C})) \simeq \mathbb{L}^* \oplus \mathbb{L}^*$. Take any closed point $p \in X$ and set $U = X \setminus \{p\}$ (which is affine). We get the localization diagram

$$(9.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{\mathrm{alg}}^*(\{p\}) & \longrightarrow & \Omega_{\mathrm{alg}}^*(X) & \longrightarrow & \Omega_{\mathrm{alg}}^*(U) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{L}^* & \longrightarrow & \mathbb{L}^* \oplus \mathbb{L}^* & \longrightarrow & \mathbb{L}^* \longrightarrow 0, \end{array}$$

where the bottom exact row is the sequence of MU^{2*} groups of the spaces $\{p\}$, X and U . The top row is exact because the left vertical map is an isomorphism (plus Theorem 6.5). The right vertical map is an isomorphism because U is affine. Hence, the middle map is an isomorphism too. \square

As an immediate corollary of Theorems 8.4 and 9.6, we obtain the following analogue of Quillen-Lichtenbaum conjecture for the cobordism of smooth curves.

Corollary 9.7. *For a smooth curve X over \mathbb{C} and an integer $m \geq 1$, the natural map $\Omega^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m \rightarrow \mathrm{MU}^{2*}(X(\mathbb{C})) \otimes_{\mathbb{Z}} \mathbb{Z}/m$ is an isomorphism.*

9.4. Surfaces. For an algebraic surface X , the Chow groups of 1-cycles as well as 0-cycles modulo rational equivalence often form infinitely generated abelian groups. Since the algebraic cobordism contains more data than Chow groups as shown in [12, Theorem 4.5.1], the algebraic cobordism of a surface is often infinitely generated as an \mathbb{L}^* -module. However, under algebraic equivalence, the algebraic cycles on an algebraic surface always form a finitely generated group, by Néron-Severi theorem. We prove an analogous result for the \mathbb{L}^* -module $\Omega_{\mathrm{alg}}^*(X)$. We use the following graded Nakayama lemma whose proof is an elementary application of a backward induction argument. It is left as an exercise.

Lemma 9.8. *Let M^* be a \mathbb{Z} -graded \mathbb{L}^* -module such that for some integer $N \geq 0$, we have $M^n = 0$ for all $n > N$. Suppose that $S = \{\alpha_1, \dots, \alpha_r\}$ is a set of homogeneous elements in $M^{\geq 0}$ whose images generate $M^* \otimes_{\mathbb{L}^*} \mathbb{Z}$ as an abelian group. Then M^* is generated by S as an \mathbb{L}^* -module.*

Theorem 9.9. *Let X be a connected smooth projective surface. Then $\Omega_{\mathrm{alg}}^*(X)$ is a finitely generated \mathbb{L}^* -module with at most $\rho + 2$ generators, where ρ is the minimal number of generators of the Néron-Severi group $\mathrm{NS}(X)$.*

Note that if $\mathrm{NS}(X)$ is torsion free, then ρ is the Picard number of X .

Proof. This follows immediately from Theorem 8.1 and Lemma 9.8, using the fact that $\mathrm{CH}_{\mathrm{alg}}^*(X) \simeq \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$. \square

9.5. Threefolds and beyond. We saw that for a smooth projective scheme X of dimension ≤ 2 , the \mathbb{L}^* -module $\Omega_{\mathrm{alg}}^*(X)$ is finitely generated. But, this is the highest we can go. This is due to the following result and some known deep results about algebraic cycles. Recall that for a smooth projective complex scheme X , the Griffiths group $\mathrm{Griff}^r(X)$ of X is the group of codimension r homologically trivial cycles modulo algebraic equivalence. In particular, it is a subgroup of $\mathrm{CH}_{\mathrm{alg}}^r(X)$.

Theorem 9.10. *For any smooth scheme X , the following two statements are equivalent:*

- (1) *The Chow group $\mathrm{CH}_{\mathrm{alg}}^*(X)$ modulo algebraic equivalence is finitely generated.*
- (2) *The cobordism $\Omega_{\mathrm{alg}}^*(X)$ is a finitely generated \mathbb{L}^* -module.*

If X is a smooth projective complex variety, then the following statement is also equivalent to the above two:

(3) The Griffiths group $\text{Griff}^*(X)$ is finitely generated.

Proof. The equivalence (1) \Leftrightarrow (2) follows by applying Theorem 8.1 and Lemma 9.8 to $M^* = \Omega_{\text{alg}}^*(X)$.

When X is a smooth projective complex variety, let $\text{CH}_{\text{hom}}^*(X)$ denote the group of algebraic cycles on X modulo homological equivalence. The equivalence (1) \Leftrightarrow (3) follows from the exact sequence

$$(9.3) \quad 0 \rightarrow \text{Griff}^*(X) \rightarrow \text{CH}_{\text{alg}}^*(X) \rightarrow \text{CH}_{\text{hom}}^*(X) \rightarrow 0$$

and the observation that $\text{CH}_{\text{hom}}^*(X)$ is a subgroup of $H^{2*}(X(\mathbb{C}), \mathbb{Z})$, which is a finitely generated abelian group since X is smooth and projective. \square

Remark 9.11. It was shown by Griffiths [7] that the Griffiths groups can be nontrivial. Clemens [1] later showed that $\text{Griff}^2(X)$ is not finitely generated for a general quintic threefold X . These results were generalized by Nori [15] for algebraic cycles of codimension ≥ 2 . Thus, it follows from Theorem 9.10 that the \mathbb{L}^* -module $\Omega_{\text{alg}}^*(X)$ is in general not finitely generated for a variety of dimension at least three.

It seems that certain questions about algebraic cycles of smooth projective schemes can be lifted to the level of cobordism cycles. As an example, consider the following. We saw in Section 9.1 that for a smooth complex variety X , there are cycle class maps $\theta_X: \Omega^*(X) \rightarrow \text{MU}^{2*}(X(\mathbb{C}))$ and $\theta_X^{\text{alg}}: \Omega_{\text{alg}}^*(X) \rightarrow \text{MU}^{2*}(X(\mathbb{C}))$. Let $\Phi_X: \Omega^*(X) \rightarrow \Omega_{\text{alg}}^*(X)$ be the natural map. We define the *Griffiths groups for the cobordism cycles* to be the graded group

$$(9.4) \quad \text{Griff}_{\Omega}^*(X) = \ker(\theta_X) / \ker(\Phi_X).$$

The subgroup $\ker(\theta_X)$ can be called the group of cobordism cycles *homologically equivalent to zero*. We ask the following:

Question 9.12. *Let X be a smooth projective complex variety of dimension at least three. Is it true that $\text{Griff}_{\Omega}^*(X)$ is a finitely generated \mathbb{L}^* -module if and only if $\text{Griff}^*(X)$ is a finitely generated abelian group? In particular, are there examples where $\text{Griff}_{\Omega}^*(X)$ is not finitely generated as an \mathbb{L}^* -module?*

10. RATIONAL SMASH-NILPOTENCE FOR COBORDISM

It was proven by Voevodsky [20] and Voisin [21] that if an algebraic cycle α on a smooth projective scheme X is zero in $\text{CH}_{\text{alg}}^*(X)_{\mathbb{Q}}$, then the smash-product $\alpha^{\otimes N} := \alpha \times \cdots \times \alpha$ on $X^N := X \times \cdots \times X$ is zero in $\text{CH}_*(X^N)_{\mathbb{Q}}$ for some integer $N > 0$. We use the notation $\alpha^{\otimes N}$ instead of α^N . The latter symbol denotes the self-intersection of α in $\text{CH}_*(X)_{\mathbb{Q}}$. This section studies the corresponding question for cobordism cycles.

Definition 10.1. Let $X \in \mathbf{Sch}_k$ and let $\alpha \in \mathcal{Z}_*(X)$. Let $N \geq 1$ be an integer.

(1) The N -fold smash-product $\alpha^{\boxtimes N} \in \mathcal{Z}_*(X^N)$ is the N -fold self-external product $\alpha \times \cdots \times \alpha$ (see Definition 2.2).

(2) α is *rationally smash-nilpotent*, if there is an integer $N > 0$ such that the image of $\alpha^{\boxtimes N}$ in $\Omega_*(X^N)_{\mathbb{Q}}$ is zero.

Lemma 10.2. *Let $X \in \mathbf{Sch}_k$ and let $\alpha, \beta \in \mathcal{Z}_*(X)$.*

(1) *If α or β is rationally smash-nilpotent, then so is $\alpha \times \beta$.*

(2) *If α and β are rationally smash-nilpotent, then so is $\alpha + \beta$.*

Proof. Note that the external product \times is commutative because in Definition 2.1 we identified all isomorphic cobordism cycles. For (1), if $\alpha^{\boxtimes N} = 0 \in \Omega_*(X^N)_{\mathbb{Q}}$, then $(\alpha \times \beta)^{\boxtimes N} = \alpha^{\boxtimes N} \times \beta^{\boxtimes N} = 0 \in \Omega_*(X^{2N})_{\mathbb{Q}}$. The case $\beta^{\boxtimes N} = 0$ in $\Omega_*(X^N)_{\mathbb{Q}}$ is similar. For (2), use the binomial theorem since \times is commutative. \square

We now prove the cobordism analogue of the result [20, Corollary 3.2]:

Theorem 10.3. *Let X be a smooth projective scheme and let $\alpha \in \mathcal{Z}_*(X)$. If the image of α in $\Omega_*^{\text{alg}}(X)_{\mathbb{Q}}$ is trivial, then it is rationally smash-nilpotent.*

Proof. By [13, Theorem 1] and Theorem 6.2, we may identify $\Omega_*(X)$ and $\Omega_*^{\text{alg}}(X)$ with $\omega_*(X)$ and $\omega_*^{\text{alg}}(X)$, respectively. Consider the exact sequence of Theorem 5.1 with rational coefficients,

$$\bigoplus_{(C, t_1, t_2)} \omega_*(X \times C)_{\mathbb{Q}} \xrightarrow{\theta'} \omega_*(X)_{\mathbb{Q}} \xrightarrow{\Psi_X} \omega_*^{\text{alg}}(X)_{\mathbb{Q}} \rightarrow 0$$

and look at the image of α in $\omega_*(X)_{\mathbb{Q}}$, also denoted by α . Since α belongs to $\ker \Psi_X$ by the given assumption, we have $\alpha \in \text{Im}(\theta')$. By Lemma 10.2-(2), it is enough to consider α of the form $(i_1^* - i_2^*)(\beta)$ for $\beta = [g: Y \rightarrow X \times C] \in \omega_*(X \times C)$. So, we suppose $\alpha = (i_1^* - i_2^*)(\beta)$.

Since $X \times C$ is smooth, by the transversality [12, Proposition 3.3.1] combined with [13, Theorem 1], we may assume that g is transversal to the closed immersions i_j , $j = 1, 2$. Hence, the fiber product Y_{t_j} of $X \times \{t_j\}$ and Y over $X \times C$ is smooth and $i_j^*(\beta) = [Y_{t_j} \rightarrow X]$. On the other hand, if we let $\pi := pr_2 \circ g: Y \rightarrow X \times C \rightarrow C$ (which is projective), we see that $\pi^*([\{t_j\} \rightarrow C]) = [Y_{t_j} \rightarrow Y]$. Hence, for $f := pr_1 \circ g: Y \rightarrow X \times C \rightarrow X$ (which is projective because C is projective), we get $f_*\pi^*([\{t_j\} \rightarrow C]) = [Y_{t_j} \rightarrow X]$ so that $(i_1^* - i_2^*)(\beta) = [Y_{t_1} \rightarrow X] - [Y_{t_2} \rightarrow X] = f_*\pi^*([\{t_1\} \rightarrow C] - [\{t_2\} \rightarrow C])$. Set $\gamma := [\{t_1\} \rightarrow C] - [\{t_2\} \rightarrow C] \in \omega_0(C)_{\mathbb{Q}}$.

We then have $\alpha = f_*\pi^*(\gamma)$ with $\gamma \in \omega_0(C)_{\mathbb{Q}}$ such that $\gamma = 0 \in \omega_0^{\text{alg}}(C)_{\mathbb{Q}}$. We claim that γ is rationally smash-nilpotent.

Under the isomorphism $\omega_0(C)_{\mathbb{Q}} \xrightarrow{\sim} \text{CH}_0(C)_{\mathbb{Q}}$ of [12, Lemma 4.5.10], the image of γ in $\text{CH}_0(C)_{\mathbb{Q}}$ is the 0-cycle $\bar{\gamma} = [\{t_1\}] - [\{t_2\}] \in \text{CH}_0(C)_{\mathbb{Q}}$, whose image in $\text{CH}_0^{\text{alg}}(C)_{\mathbb{Q}}$ is trivial. Hence by [20, Corollary 3.2], we see that $\bar{\gamma}^{\otimes N} = 0 \in \text{CH}_0(C^N)_{\mathbb{Q}}$ for some integer $N > 0$. Since the isomorphism $\omega_0(C^N)_{\mathbb{Q}} \simeq \text{CH}_0(C^N)_{\mathbb{Q}}$ of [12, Lemma 4.5.10] respects the external products, we conclude that $\gamma^{\boxtimes N} = 0 \in \omega_0(C^N)_{\mathbb{Q}}$.

Since γ is rationally smash-nilpotent, we now easily see that $\alpha = f_*\pi^*(\gamma)$ is also rationally smash-nilpotent since the push-forward and the pull-back maps respect external products (cf. Theorem 7.6). \square

Remark 10.4. We remark that the proof of Theorem 10.3 uses [20] only for smooth projective curves.

Remark 10.5. Theorem 10.3 shows that all algebraically trivial cobordism cycles on smooth projective schemes are smash-nilpotent with the \mathbb{Q} -coefficients. Motivated by [20, Conjecture 4.2], one can further ask whether numerical triviality of cobordism cycles is equivalent to smash-nilpotence for a suitable notion of numerical equivalence for cobordism cycles. The authors do not know how to answer this.

As a weaker version, we wonder if homologically trivial cobordism cycles are smash-nilpotent, where homological equivalence on cobordism cycles was defined around (9.4). For abelian varieties one might try the following, motivated by [9]. Let A be an abelian variety and for each $m \in \mathbb{Z}$, let $\langle m \rangle: A \rightarrow A$ be the multiplication morphism by m . Let's call a cobordism cycle $\beta \in \Omega^*(A)_{\mathbb{Q}}$ *skew* if $\langle -1 \rangle^*(\beta) = -\beta$. A skew cobordism cycle is

homologically trivial. Our guess is that any skew cobordism cycle on an abelian variety A is smash-nilpotent. The corresponding question for algebraic cycles was answered in [9, Proposition 1] and it was deduced that any homologically trivial cycle on an abelian variety of dimension ≤ 3 is smash-nilpotent.

To answer it, the following strategy is likely to work. Firstly, use the category of cobordism motives in the sense of [14, §5.1] and [19, §2], a cobordism analogue of the category of Chow motives. Secondly, use the abelian structure on A to prove an analogue of Chow-Künneth decomposition. Lastly, imitate the arguments of [9]. A detailed discussion of this approach will appear in a separate paper.

11. APPENDIX

This section gives a summary of the constructions from [12, §6] related to the Gysin maps and the pull-backs via l.c.i. morphisms on the algebraic cobordism, that are used in this paper. The only new result is Lemma 11.6, used in the construction of the map $i_D^*: \Omega_*^{\text{alg}}(X) \rightarrow \Omega_{*-1}^{\text{alg}}(|D|)$ of (7.2) for a pseudo-divisor D on X .

Definition 11.1 ([12, 6.1.2]). Let $X \in \mathbf{Sch}_k$ and let D be a pseudo-divisor on X .

(1) $\mathcal{Z}_*(X)_D$ is the subgroup of $\mathcal{Z}_*(X)$ generated by the cobordism cycles $[f: Y \rightarrow X, L_1, \dots, L_r]$ for which either $f(Y) \subset |D|$ holds, or $f(Y) \not\subset |D|$ and $\text{Div } f^*D$ is a strict normal crossing divisor on Y .

(2) Let $\mathcal{R}_*^{\text{Dim}}(X)_D$ be the subgroup of $\mathcal{Z}(X)_D$ generated by the cobordism cycles of the form $[f: Y \rightarrow X, \pi^*(L_1), \dots, \pi^*(L_r), M_1, \dots, M_s]$, where $\pi: Y \rightarrow Z$ is smooth quasi-projective, $Z \in \mathbf{Sm}_k$, and $L_1, \dots, L_{r > \dim Z}$ are line bundles on Z . We let $\underline{\mathcal{Z}}_*(X)_D := \mathcal{Z}_*(X)_D / \mathcal{R}_*^{\text{Dim}}(X)_D$.

The projective push-forward and smooth pull-back on $\mathcal{Z}_*(-)_D$ can be defined as for $\mathcal{Z}_*(-)$, and likewise for $\underline{\mathcal{Z}}_*(-)$.

(3) For a line bundle $L \rightarrow X$, define the Chern class operation $\tilde{c}_1(L): \mathcal{Z}_*(X)_D \rightarrow \mathcal{Z}_{*-1}(X)_D$ as for $\mathcal{Z}_*(X)$. This descends onto $\underline{\mathcal{Z}}_*(X)_D$.

(4) We have the external product

$$\times: \mathcal{Z}_*(X)_D \otimes \mathcal{Z}_*(X')_{D'} \rightarrow \mathcal{Z}_*(X \times X')_{pr_1^*D + pr_2^*D'}$$

as for $\mathcal{Z}_*(-)$. This descends onto $\underline{\mathcal{Z}}_*(-)_D$ -level.

Let $X \in \mathbf{Sch}_k$, D be a pseudo-divisor on X , and f be a projective morphism $f: Y \rightarrow X$ from a smooth irreducible scheme Y . A strict normal crossing divisor E on Y is said to be in *very good position with D* if either $f(Y) \subset |D|$ holds, or $f(Y) \not\subset |D|$ and $E + \text{Div } f^*D$ is a strict normal crossing divisor on Y . If E is in very good position with D , then for each face $i_J: E_J \hookrightarrow E$ and the induced composition $f_J := f \circ i_J: E_J \rightarrow Y \rightarrow X$, either $f_J(E_J) \subset |D|$ holds, or $\text{Div } f_J^*D$ is a strict normal crossing divisor on E_J , by [12, Remark 6.1.4(1)]

Definition 11.2 ([12, Definition 6.1.5]). Let $X \in \mathbf{Sch}_k$ and let D be a pseudo-divisor on X . Let $\mathcal{R}_*^{\text{Sect}}(X)_D$ be the subgroup of $\mathcal{Z}_*(X)_D$ generated by elements of the form $[f: Y \rightarrow X, L_1, \dots, L_r] - [f \circ i: Z \rightarrow X, i^*(L_1), \dots, i^*(L_{r-1})]$, with $r > 0$, such that

(1) $[f: Y \rightarrow X, L_1, \dots, L_r] \in \mathcal{Z}_*(X)_D$, and

(2) $i: Z \rightarrow Y$ is the closed immersion of the subscheme given by the vanishing of a transverse section $s: Y \rightarrow L_r$ such that Z is in very good position with D .

We let $\underline{\Omega}_*(X)_D := \underline{\mathcal{Z}}_*(X)_D / \mathcal{R}_*^{\text{Sect}}(X)_D$.

Definition 11.3 ([12, Definitions 6.1.6]). Let $X \in \mathbf{Sch}_k$ and let D be a pseudo-divisor on X .

(1) Let $\mathcal{R}_*(X)_D$ be the subgroup of $\mathcal{Z}_*(X)_D$ generated by elements of the form $[Y \rightarrow X, L_1, \dots, L_r] - [Y' \rightarrow X, L'_1, \dots, L'_r]$ such that

- (a) $[Y \rightarrow X, L_1, \dots, L_r]$ and $[Y' \rightarrow X, L'_1, \dots, L'_r]$ are in $\mathcal{Z}_*(X)_D$, and
- (b) there exist an isomorphism $\phi: Y \rightarrow Y'$ over X , a permutation σ of $\{1, \dots, r\}$ and an isomorphism $L_i \simeq \phi^*(L'_{\sigma(i)})$.

We define $\underline{\Omega}_*(X)_D := \underline{\Omega}_*(X)_D / \mathcal{R}_*(X)_D$.

(2) Let $\Omega_*(X)_D$ be the quotient of $\mathbb{L}_* \otimes_{\mathbb{Z}} \underline{\Omega}_*(X)_D$ by the relations of the form

$$(\text{Id}_{\mathbb{L}_*} \otimes f_*)(F_{\mathbb{L}_*}(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) - \tilde{c}_1(L \otimes M)(\eta))$$

where L, M are line bundles on Y , $\eta \in \underline{\Omega}_*(Y)_D$ and $[f: Y \rightarrow X]$ is a cobordism cycle for which either $f(Y) \subset |D|$ holds, or $f(Y) \not\subset |D|$ and $\text{Div } f^*D$ is a strict normal crossing divisor on Y .

The Chern class operation and the external product are induced on $\Omega_*(-)_D$.

It is clear from the above definition that there is a natural map $\theta_X: \Omega_*(X)_D \rightarrow \Omega_*(X)$. The main content of [12, §6.4.1] is the proof of the following moving lemma.

Theorem 11.4 ([12, Theorem 6.4.12]). *For $X \in \mathbf{Sch}_k$, the natural map $\theta_X: \Omega_*(X)_D \rightarrow \Omega_*(X)$ is an isomorphism.*

Now we define the intersection by D on $\Omega_*(X)_D$, namely, $D(-): \Omega_*(X)_D \rightarrow \Omega_{*-1}(|D|)$. First recall the map $D(-): \mathcal{Z}_*(X)_D \rightarrow \Omega_{*-1}(|D|)$.

Definition 11.5 ([12, §6.2.1]). Let $X \in \mathbf{Sch}_k$ and let $D = (|D|, O_X(D), s)$ be a pseudo-divisor on X . Let $\eta := [f: Y \rightarrow X, L_1, \dots, L_r] \in \mathcal{Z}_*(X)_D$.

(1) If $f(Y) \subset |D|$, let $f^D: Y \rightarrow |D|$ be the induced morphism from f . Note that $\tilde{c}_1(f^*O_X(D))([\text{Id}_Y: Y \rightarrow Y, L_1, \dots, L_r]) \in \Omega_{*-1}(Y)$. We define

$$D(\eta) := f_*^D \{ \tilde{c}_1(f^*O_X(D))([\text{Id}_Y: Y \rightarrow Y, L_1, \dots, L_r]) \} \in \Omega_{*-1}(|D|).$$

(2) If $f(Y) \not\subset |D|$, then $\tilde{D} := \text{Div } f^*D$ is a strict normal crossing divisor on Y . Let $f^D: |\tilde{D}| \rightarrow |D|$ be the restriction of f , and L_i^D be the restriction of L_i on $|\tilde{D}|$. We define $D(\eta) := f_*^D \{ \tilde{c}_1(L_1^D) \circ \dots \circ \tilde{c}_1(L_r^D)([\tilde{D} \rightarrow |\tilde{D}|]) \} \in \Omega_{*-1}(|D|)$, where the cobordism cycle $[\tilde{D} \rightarrow |\tilde{D}|] \in \Omega_*(|\tilde{D}|)$ is discussed in Section 5.1 and [12, §3.1].

This descends to give $D(-): \Omega_*(X)_D \rightarrow \Omega_{*-1}(|D|)$ by [12, §6.2]. To show that it induces (7.2), we need the following:

Lemma 11.6. *Let $X \in \mathbf{Sch}_k$ and let D be a pseudo-divisor on X . Let $[f: Y \rightarrow X, L]$ and $[f: Y \rightarrow X, M]$ be two cobordism cycles in $\Omega_*(X)$ such that $L \sim M$. Let $\eta := [Y \rightarrow X, L] - [Y \rightarrow X, M]$. Then $D \circ \phi_X(\eta) \in \ker(\Omega_{*-1}(|D|) \rightarrow \Omega_{*-1}^{\text{alg}}(|D|))$, where $\phi_X = \theta_X^{-1}$ of Theorem 11.4.*

Proof. We first assume that $[f: Y \rightarrow X, L]$ and $[f: Y \rightarrow X, M]$ lie in $\Omega_*(X)_D$ and show that $D(\eta) \in \ker(\Omega_{*-1}(|D|) \rightarrow \Omega_{*-1}^{\text{alg}}(|D|))$. By the definition of $D(-)$ in Definition 11.5, there is nothing to prove if $f(Y) \subset |D|$. So, suppose $f(Y) \not\subset |D|$. Then, we have

$$(11.1) \quad D([f: Y \rightarrow X, L]) = f_*^D \{ \tilde{c}_1(L|_{\tilde{D}})([\tilde{D} \rightarrow |D|]) \} \in \Omega_{*-1}(|D|)$$

and the similar expression holds for $D([f: Y \rightarrow X, M])$. On the other hand, $L \sim M$ implies that $L|_{\tilde{D}} \sim M|_{\tilde{D}}$ and hence $\tilde{c}_1(L|_{\tilde{D}}) = \tilde{c}_1(M|_{\tilde{D}})$ as operators on $\Omega_{*-1}^{\text{alg}}(|\tilde{D}|)$. Using Proposition 3.10 and applying f_*^D , we get from (11.1) that $D([f: Y \rightarrow X, L]) = D([f: Y \rightarrow X, M])$ in $\Omega_{*-1}^{\text{alg}}(|D|)$. Equivalently, $D(\eta) \in \ker(\Omega_{*-1}(|D|) \rightarrow \Omega_{*-1}^{\text{alg}}(|D|))$.

To complete the proof, choose a suitable projective birational map $\rho: W \rightarrow Y \times \mathbb{P}^1$ as in [12, Lemma 6.4.1]. This yields a commutative diagram

$$(11.2) \quad \begin{array}{ccc} \Omega_*(W)_{D_W} & \xrightarrow{D_W(-)} & \Omega_{*-1}(|D_W|) \\ (f \circ pr_1 \circ \rho)_* \downarrow & & \downarrow (f \circ pr_1 \circ \rho)_* \\ \Omega_*(X)_D & \xrightarrow{D(-)} & \Omega_{*-1}(|D|) \end{array}$$

where $D_W = \rho^* \circ pr_1^* \circ f^*(D)$ such that

$$D \circ \phi_X([Y \rightarrow X, L]) = (f \circ pr_1 \circ \rho)_* \circ D_W([\rho^*(Y \times \{0\}) \rightarrow W, \rho^*(L)]).$$

A similar formula holds for $D \circ \phi_X([Y \rightarrow X, M])$. The lemma follows from this by applying what we have shown above to the pair (W, D_W) in place of (X, D) . \square

Acknowledgments. The authors feel very grateful to Marc Levine and the anonymous referee of the Journal of K -theory for useful comments on various parts of this work and for suggesting further research directions.

JP would like to thank Juya and Dany for the constant supports and peace of mind at home during the work. He also wishes to acknowledge that part of this work was written during his visits to TIFR and the Universität Duisburg-Essen in 2011. He would like to thank all those institutions that made this work possible.

During this work, JP was partially supported by the National Research Foundation of Korea (NRF) grant (No. 2011-0001182) and Korea Institute for Advanced Study (KIAS) grant, both funded by the Korean government (MEST), and the TJ Park Junior Faculty Fellowship funded by POSCO TJ Park Foundation.

REFERENCES

- [1] H. Clemens, Homological equivalence, modulo algebraic equivalence is not finitely generated, *Pub. Math. IHES* 58 (1983) 19–38.
- [2] S. Dai, Algebraic cobordism and Grothendieck groups over singular schemes, *Homology, Homotopy Appl.* 12 (2010) 93–110.
- [3] E. Friedlander and H. B. Lawson, A theory of algebraic cocycles, *Ann. of Math.* 136 (1992) 361–428.
- [4] E. Friedlander and M. E. Walker, Comparing K -theories for complex varieties, *Amer. J. Math.* 123 (2001) 779–810.
- [5] E. Friedlander and M. E. Walker, Semi-topological K -theory using function complexes, *Topology* 41 (2002) 591–644.
- [6] W. Fulton, Intersection theory, Second Edition, *Ergebnisse der Math. Grenzgebiete* 3 (Springer-Verlag, Berlin, 1998). xiv+470 pp.
- [7] P. Griffiths, On the periods of certain rational integrals II, *Ann. of Math.* 90 (1969) 496–541.
- [8] J. Hornbostel and V. Kiritchenko, Schubert calculus for algebraic cobordism, *J. Reine Angew. Math.* 656 (2011), 59–85.
- [9] B. Kahn and R. Sebastian, Smash-nilpotent cycles on Abelian 3-folds, *Math. Res. Lett.* 16 (2009) no. 6, 1007–1010.
- [10] S. Kleiman and A. Altman, Bertini theorems for hypersurface sections containing a subscheme, *Comm. Algebra* 7 (1979) 775–790.
- [11] A. Krishna, Equivariant cobordism for torus actions, arXiv:1010.6182v1 [math.AG]
- [12] M. Levine and F. Morel, Algebraic Cobordism, *Springer Monographs Math.* (Springer, Berlin, 2007). xii+244 pp.
- [13] M. Levine and R. Pandharipande, Algebraic cobordism revisited, *Invent. Math.* 176 (2009) 63–130.
- [14] A. Nenashev and K. Zainoulline, Oriented cohomology and motivic decompositions of relative cellular spaces, *J. Pure Appl. Algebra* 205 (2006), 323–340.
- [15] M. Nori, Algebraic cycles and Hodge-theoretic connectivity, *Invent. Math.* 111 (1993) 349–373.
- [16] D. Quillen, Elementary proofs of some results of cobordism theory using Steenrod operations, *Adv. Math.* 7 (1971), 29–56.

- [17] P. Samuel, Relations d'équivalence en géométrie algébrique, *Proc. Internat. Congress Math. 1958* (Cambridge Univ. Press, New York, 1960) 470–487.
- [18] B. Totaro, Torsion algebraic cycles and complex cobordism, *J. Amer. Math. Soc.* 10 (1997) 467–493.
- [19] A. Vishik and N. Yagita, Algebraic cobordisms of a Pfister quadric, *J. London Math. Soc.* 76 (2007) no. 2, 586–604.
- [20] V. Voevodsky, A nilpotence theorem for cycles algebraically equivalent to zero, *Internat. Math. Res. Notices* (1995) 187–198.
- [21] C. Voisin, Remarks on zero-cycles of self-products of varieties, in Maruyama, Masaki (ed.), *Moduli of vector bundles, Lect. Notes in Pure Appl. Math.* 179 (Marcel Dekker, New York, 1996) 265–285.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, 1 HOMI BHABHA ROAD, COLABA, MUMBAI, 400 005, INDIA

E-mail address: amal@math.tifr.res.in

DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, 291 DAEHAK-RO, YUSEONG-GU, DAEJEON, 305-701, REPUBLIC OF KOREA (SOUTH)

E-mail address: jinhyun@mathsci.kaist.ac.kr; jinhyun@kaist.edu